Exercise 10

1. This problem is about the discretization of parabolic PDEs. Consider the variational equation

Find $u : [0,T] \to H_0^1(D)$, s.t. $u \in L^2(0,T,H_0^1(D)) \cap C^0([0,T],L^2(D))$, and

$$\frac{\partial}{\partial t}\int_D u(x,t)v(x)\,dx + \underbrace{\int_D \nabla u(x,t)\nabla v(x)\,dx}_{=:a(u(x,t),v)} = \underbrace{\int_D f(x,t)v(x)\,dx}_{=:l(t;v)} \quad \forall v \in V, \qquad (1)$$
$$u(0) \equiv u_0,$$
$$u|_{\partial D} \equiv 0\ ,$$

where $f \in L^2(0,T,L^2(D))$ and $u_0 \in L^2(D)$ are given in advance and D is, as usual, an open bounded domain in $\mathbb{R}^2$.
Let $\mathcal{T}$ be a regular simlpicial triangulation of D.

a) As for stationary equations, we choose a finite-dimensional subspace $V_N \subseteq V$, $\dim(V_N) = N$ with basis functions $\{b_i,\ i = 1,\dots,N\}$. The Galerkin approach reformulates (1) in the V_N setting, i.e. we get

$$\frac{\partial}{\partial t}\int_D u_N(x,t)v_N(x)\,dx + a(u_N(x,t),v_N) = l(t;v_N) \quad \forall v_N \in V_N, \qquad (2)$$
$$u(0) \equiv \Pi_N u_0,$$
$$u|_{\partial D} \equiv 0\ ,$$

where $\Pi_N : L^2(D) \twoheadrightarrow V_N$ is a projection [1].
Show that this is equivalent to the ODE

$$M\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = \mathbf{l}(t), \quad \mathbf{u}(0) = \mathbf{u}_0\ ,$$

where $\mathbf{u}(t)$ is the coefficient vector of u_N at time t, $\mathbf{u}_0$ is the coefficient vector of $\Pi[u_0]$, $\mathbf{l}(t) = (l_i(t))$, with $l_i(t) = l(t;b_i)$, and M and A are the mass and stiffness matrices.

b) In order to solve this ODE numerically, choose $N \in \mathbb{N}$, $\Delta t := \frac{T}{N}$, which yields $\{t_n = n\Delta t,\ n = 0,\dots,N\}$. We search for $\mathbf{u}^{(n)} \simeq \mathbf{u}(t_n)$.

[1] Write $\Pi[v](x) = \sum v_j b_j(x)$. As for the implementation, the L^2-representant of u_0 is fixed and given by a (computable) formula, this implies that for nodal FEM, the basis coefficients $u_0^{(j)}$ are given by evaluation of u_0.

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Here, we use the ϑ scheme: Choose $\vartheta \in [0,1]$ and iterate

$$\mathbf{u}^{(0)} := \mathbf{u}_0$$
$$(M + \vartheta \Delta t A)\mathbf{u}^{(n)} = (M + (1-\vartheta)\Delta t A)\mathbf{u}^{(n)} + \Delta t((1-\vartheta)\mathbf{l}^{(n)} + \vartheta \mathbf{l}^{(n+1)}), \quad n \geq 0\ .$$

Write a matlab solver for linear FEM. That is, write a function `u=Solve_Heat_LFETheta(...)`, which takes as inputs the mesh `Mesh`, the vector `t` of times t_n, the function handles `u0=@(x)...` and `f=@(x,flag,t)...` and the iteration parameter `theta`, and returns an array `u` which contains the solution vector $\mathbf{u}^{(n)}$ in the n-th column (in matlab, indices have to start from 1, not from 0).

Hint: You can use the standard LehrFEM functions to assemble the load vector. In order to pass the time as input, make use of the `varargin` option.
Take care implementing the boundary conditions!

c) Set $D := [0,1] \times [0,2]$. Set up an initial mesh with meshwidth .5 on D.
Set $f(x,t) = \chi_{B_{0.25}((0.5,1))}(x)$, $u_0(x) = 0$, where $B_r(p)$ is the disc of radius r centred at $p \in D$.
Solve the equation (1) with linear FEM and the ϑ method, with $\vartheta = 0, 1$. For each time, plot the solution in the same scale, such that you get a movie. What do you observe, and why?

d) (optional) In order to create a movie in matlab, the combination of the functions `getframe()` and `movie` is useful. Inform yourself about these functions and try to construct a movie in matlab. In order to save it in a format which is playable outside matlab (i.e. `.avi` or even an animated `.gif`), there is a function `movie2avi.m` already implemented, or for gif-files, you can download `movie2gif.m` from the matlab central[2].

2. a) In the notation of the existence and uniqueness theorem seen in the lecture, show that

$$u_m(t) := \sum_{1}^{m} (u_0, w_i) e^{-\lambda_i t} + \int_0^t (f(s), w_i) e^{-\lambda_i (t-s)}\, ds$$

is a Cauchy sequence in both $L^2(0,T;V)$ and $C^0([0,T];H)$.

b) Conclude existence and uniqueness of solutions to the general framework of Chapter 3.2 for $f \not\equiv 0$.

3. Let D be the space occupied by a (physical) body with heat conductivity $\varkappa(x)$. At $t = 0$, we observe a heat distribution $u_0(x)$ and observe the evolution of the heat distribution $u(x,t)$ in the observation period $t \in [0,T]$ under the influence of a source $f : D \times [0,T] \to \mathbb{R}$ and a surface

[2] http://www.mathworks.com/matlabcentral/fileexchange/17463-movie-to-gif-converter

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temperature $g : \partial D \times [0,T] \to \mathbb{R}$.
Fourier's law (obtained from experimental evidence) states that the heat flux vectorfield $\mathbf{j}$ can be obtained as $\mathbf{j}(x,t) = -\varkappa(x)\nabla u(x,t)$.
Moreover, we assume energy conservation, i.e. for all control volumes $V \subseteq D$,

$$\frac{\partial}{\partial t}\int_V u(x,t)\,dx + \int_{\partial V} \mathbf{j}(x,t)\cdot\boldsymbol{\nu}(x)\,ds = \int_V f(x,t)\,dx\ ,$$

where $\boldsymbol{\nu}$ denotes the outward pointing unit normal vector.
Prove that for all $(x,t) \in D \times [0,T]$.

$$\frac{\partial}{\partial t}u(x,t) - \nabla\cdot(\varkappa(x)\nabla u(x,t)) = f(x,t)\ .$$

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Dec 2**, 2013