Exercise 11

1. Consider the ordinary differential equation

$$M\dot{y}(t) + Ay(t) = l(t), \quad y(t^{(0)}) = y_0 \,.$$

Find the consistency order of the *SDIRK-2* scheme for this equation.
The SDIRK[1]-2 scheme is given as Runge-Kutta scheme with the Butcher tableau

$$\begin{array}{c|cc} \lambda & \lambda & 0 \\ 1 & 1-\lambda & \lambda \\ \hline & 1-\lambda & \lambda \end{array} \quad ,$$

where $\lambda = 1 - \frac{\sqrt{2}}{2}$.
Moreover, the *consistency order* of a scheme which yields discrete solutions $y^{(n)} \simeq y(t^{(n)})$ is the largest integer $p \in \mathbb{N}$, such that $\|y(t^{(0)} + \Delta t) - y^{(1)}\| = O(\Delta t^p)$, for $\Delta t \to 0$.

2. Let $\rho > 0$ and $\mu > 0$ be constants. We consider the following system of PDEs in two spatial and one time dimensions, with solutions $(\mathbf{u}(x,t), p(x,t))$, where $\mathbf{u} = (u_1, u_2)$, and a right hand side $\mathbf{f} = (f_1, f_2)$. This system is called the *(nonstationary) Stokes problem*:

$$\begin{aligned} \rho\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mu \operatorname{grad} p &= \mathbf{f}, \quad \text{in } D \times (0,T), \text{ for } i = 1,2, \\ \operatorname{div} \mathbf{u} &= 0, \quad \text{in } D \times (0,T) \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \\ \mathbf{u}(x,t) &= 0, \quad \forall\, (x,t) \in \partial D \times (0,T) \,, \end{aligned} \tag{1}$$

where in the first equation, the Laplace operator Δ applied to the vector field $\mathbf{u}$ has to be interpreted componentwise.
We define the spaces $J_0 := \{\mathbf{v} \in H_0^1(D)^2 \;:\; \operatorname{div}(\mathbf{v}) = 0 \text{ in } L^2(D)\}$, and $H := \overline{J_0}^{\|\,\|_{L^2(D)}}$, i.e. the completion of J_0 with respect to the L^2-norm.
Moreover, let $a(\mathbf{u}, \mathbf{v}) := \mu \sum_{i,j=1}^2 \int_D \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx$ and recall the definition of the scalar product for L^2-spaces of vector fields: $(\mathbf{u}, \mathbf{v}) := \sum_{i,j=1}^2 \int_D u_i v_i \, dx$.

a) Inform yourself about the physical situation this system is modelling.

[1]"singly diagonally implicit Runge-Kutta"

b) Derive a variational form of (1) using J_0, H, a and integration by parts.

c) Prove that for all $\mathbf{u}_0 \in H$ and $f \in L^2(0,T;L^2(D))$, this variational form possesses a unique solution $\mathbf{u} \in L^2(0,T;J_0) \cap C^0([0,T];H)$.

d) The pressure p can be retrieved solving an additional equation, the *pressure Poisson equation*. To derive this PDE, apply the divergence operator on the first line of (1). Describe two issues when retrieving p this way.

3. Consider the semidiscrete heat equation

$$M\dot{u}(t) + Au(t) = 0, \quad u(t^{(0)}) = y_0 .$$

Suppose, M and A arose from a discretization with piecewise linear FEM. Discretization by explicit Euler yields

$$My^{(n)} = (M - \Delta t A)y^{(n-1)}.$$

a) From there, derive a condition on Δt which guarantees stability of the scheme which is of the form "The scheme is stable if $\Delta t <= \Delta t_0$, where Δt_0 is determined by the eigenvalues of some matrix M_1".

b) In matlab, write a script `Stab_LFE_EE.m` which constructs an initial mesh on the unit square $(0,1)^2$ of size $h_0 := 0.2$, and for 5 regular refinements (the initial mesh being the first one) it assembles M, A and computes Δt_0.
Plot Δt_0 versus h into a loglog plot. Does your experiment suggest that there is an algebraic relation of the form $\Delta t_0 = O(h^q)$? If yes, what q do you get?
Hint: If there is a M^{-1} in your definition of M_1, reformulate the eigenvalue problem for M_1 as a generalized eigenvalue problem of the form $M_2 v = \lambda M v$.
The command `eigs` can compute the desired m largest or smallest eigenvalues or generalized eigenvalues of a sparse matrix M_1 or a pair of sparse matrices (M_2, M), respectively. Type `help eigs`, or use the online description under `http://www.mathworks.ch/ch/help/matlab/ref/eigs.html`.

c) Repeat the exercise for the Heun method. The Heun method is a Runge-Kutta scheme with Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 0 & 1 \end{array} .$$

d) Repeat the exercise for the SDIRK-2 method given in Exercise 1.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Dec 2**, 2013