Exercise 12

1. If you have not already done that for problem sheet 10, write a function `[u, t]=SolveHeat_LFETheta(T, dt, Mesh, u0, f, dir, theta)` that solves the heat equation with Dirichlet conditions

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= f(x,t), \quad (x,t) \in D \times (0,T) \\ u(x,t) &= \mathrm{Dir}(x,t), \quad (x,t) \in \partial D \times (0,T), \\ u(x,0) &= u_0(x), \quad x \in D\ . \end{aligned}$$

using piecewise linear FE and the ϑ-scheme[1] with parameter `theta` on the mesh `Mesh` using a timestep `dt`.

Use this solver to perform the following convergence tests with $\vartheta = 1$ (implicit Euler) and $\vartheta = \frac{1}{2}$ (Crank-Nicolson):

a) Let $D := (0,1)^2$ and $u(x,t) = \sin(\pi x_1)\sin(\pi x_2)\cos(2\pi t)$, $T = 1$, $\Delta t = 0.1$. Take an initial meshwidth $h_0 = 0.25$, refine regularly four times (leave Δt constant), and compute the L^2 and the H^1 norms of the error at the endtime $t = T$. Plot the error norms versus h into a loglog-plot. What do you observe? Why?

b) D, u, T are defined as above. Now, set up a mesh on D with constant meshwith $h = 0.05$. Compute the L^2 and H^1 error norms at endtime T for $\Delta t = 2^{-j}$, $j = 2, 3, 4, 5$. Plot the error norms versus Δt into a loglog plot. What do you observe, and why?

c) Again, D, u, T as above. For h *and* $\Delta t = \frac{1}{2^j}$, $j = 1, 2, \ldots, 5$, compute the L^2 and H^1 error norms at endtime T. Visualize it in a 3d loglog plot where h is on the x-axis, Δt on the y-axis.

Hint: There is a skeleton on the website.

2. Let $D = (0,1)$ and for $h = \frac{1}{N}$ consider the equidistant grid $\mathcal{T}_h := \{kh \ : \ k = 0, \ldots, N\}$. Let $V = H_0^1(D)$, $H = L^2(D)$ and $V_h := S_0 s^1(D, \mathcal{T}_h)$.
Compute the eigenvalues $\lambda_{k,N}$, $k = 0, \ldots, N$, of the bilinear form $a(u,v) := \int_0^1 u'(x)v'(x)\,dx$ on V_N, i.e. $a(u,v) = \lambda_{k,N}(u,v)_{L^2(D)}$.
Hint: The 1D mass and stiffness matrices can be written explicitly, see problem 1.1.c).
The continuous eigenfunctions are known to be $x \mapsto \sin(k\pi x),\ k \in \mathbb{N}$. Derive from that a guess for the k-th eigenvector. Another approach would be to study the difference equations arising from the system $Av = \lambda M v$, where A and M are in the explicit form.

[1] See problem 10.1

3. Consider the ordinary differential equation arising from the semidisretization of the heat equation with FEM of a certain polynomial degree p, as studied in the lecture:

$$M\dot{\mathbf{u}}(t) + A\mathbf{u}(t) = l(t) \text{ for } t \in (0,T], \quad \mathbf{u}(0) = u_0\ . \tag{1}$$

We want to solve this ODE using a Finite Element Method.

a) Let $0 =: t_0 < t^{(1)} < \cdots < t^{(M)} := T$ be a grid on $[0,T]$. Assume $\mathbf{u}(t^{(n)})$ to be known in advance and derive a variational formulation of (1) on the interval $(t^{(n)}, t^{(n+1)}]$, by multiplication with a test function $\mathbf{v}$.
For the trial and test spaces, assume that $\mathbf{u}$ is continuous at each t_n.

b) Examine the resulting method for the case where $\mathbf{v}$ and $\mathbf{u}$ are linear on $(t^{(n)}, t^{(n+1)}]$. Prove that the resulting scheme can be written as an iteration $M'\mathbf{u}^{(n+1)} = \ldots$, which coincides with the ϑ-scheme for a special choice of ϑ.
What is the consistency of the resulting method? (How) does this fit into our 1D FEM-theory?

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Dec 16**, 2013