Exercise 2

1. Let $D \subset \mathbb{R}^d$ be an open, bounded domain, such that it can be embedded into a cube $W := [0, s]^d$. Our aim is to prove the Poincaré-Friedrichs inequality for functions in

$$H_0^1(D) := \{v \in H^1(D) \,:\, v|_{\partial D} \equiv 0\}.$$

a) Let $u \in H_0^1(D)$. Prove that there is a constant $c > 0$ which only depends on D, such that $\|u\|_{L^2(D)} \leq c\|\nabla u\|_{L^2(D)}$.

Hint: $C_0^\infty(D) \hookrightarrow H_0^1(D)$ has a dense image[1]. Hence a good strategy is to prove the claim for $v \in C_0^\infty(D)$ and to conclude to $H^1(D)$ by a *density argument*.

The case $d = 1$ has been done on the exercise sheet 0. Find the easiest extension $\tilde{u} \in H_0^1(W)$ of u, and proceed as in $d = 1$ in each coordinate.

b) Let D be as before, and let $\Gamma \subset \partial D$ with positive measure. Adjust your proof such that the inequality holds in the space $H_\Gamma^1 := \{v \in H^1(D) \,:\, v|_\Gamma \equiv 0\}$.

2. Let $D \subset \mathbb{R}^2$ be an open, bounded domain that can be embedded into a square $[0, s]^2$. Let $\{D_k\}_{k=1,\dots,N}$ be finitely many open, bounded subsets of D with piecewise smooth ∂D_k, such that $\bar{D} = \bigcup_k \bar{D}_k$. Now, let α be an integrable function on D, such that $\alpha \in C^0(D_i)$. On $H^1(D)$, we define the bilinear form

$$a(u, v) := \int_D \nabla u(x) \nabla v(x)\, dx + \int_D \alpha(x) u(x) v(x)\, dx\ .$$

a) Find conditions on α, such that a is continuous on $H^1(D)$.

b) Find conditions on α, such that a is coercive on $H^1(D)$.

c) Prove that $a_1(u, v) := \int_D \nabla u(x) \nabla v(x)\, dx$ is not coercive on $H^1(D)$, but is indeed coercive on $H_0^1(D)$.

d) Let $l : H^1(D) \to \mathbb{R}$ be a linear, continuous functional. Under which conditions on α does the following variational problem have a unique solution?

Find $u \in H^1(D)$, such that $a(u, v) = l(v)\ \forall v \in H^1(D)$.

[1] i.e. for all $v \in H_0^1(D)$ there is a sequence $(v_k)_k$ in $C_0^\infty(D)$ s.t. $[v_k] \overset{k\to\infty}{\to} v$ in the H^1-norm, where $[v_k]$ is the H^1-class of v_k.

Please turn sheet!

3. *The inf-sup condition by Ladyzhenskaya, Babuška, and Brezzi*
Let V, W be reflexive Banach spaces. Let $a : V \times W \to \mathbb{R}$ be bilinear in $V \times W$ which is continuous, i.e.

$$\exists c_1 > 0 \ : |a(v,w)| \le c_1 \|v\|_V \, \|w\|_W \ \forall v \in V, w \in W.$$

Moreover, let $l \in W' := \{l' : W \to \mathbb{R} \ : \ l' \text{ is linear and continuous}\}$.
Now, we consider the following variational problem:
Find $u \in V$, s.t.

$$a(u,w) = l(w) \quad \forall w \in W. \tag{1}$$

There is a theorem which states that a solution $u \in V$ to this problem exists and is unique if and only if there exists a constant $c_2 > 0$, s.t.

$$\begin{aligned} &\inf_{0 \neq v \in V} \sup_{0 \neq w \in W} \frac{a(v,w)}{\|v\|_V \, \|w\|_W} \ge c_2 > 0 \ , \\ &\sup_{v \in V} a(v,w) > 0 \quad \forall 0 \neq w \in W \ . \end{aligned} \tag{2}$$

The first condition is also called the *inf-sup condition* or the *Ladyzhenskaya-Babuška-Brezzi condition*.

a) Show that if a satisfies conditions (2), there is a unique solution to (1).
Hint: The difference of this proof to the proof of Lax-Milgram (which is a special case of this theorem) is that Riesz can not be applied. Instead you have to check “Riesz” yourself:

1. For $v \in V$ arbitrary, show that there exists a unique $R(v) \in W'$ such that $R(v)(w) = a(v,w)$, and $\|R\|_{\mathcal{L}(V,W')} \le c_1$.
2. Show that $R(V) \subseteq W'$ is closed.
3. For existence of u, show that $R(V) = W'$. Recall the following corollary of Hahn-Banach: Let X be a normed vector space, $U \subseteq X$ a subspace and $x_0 \in X \setminus \bar{U}$. Then, there exists a $f \in \mathcal{L}(X, \mathbb{R})$, s.t. $\|f\|_{\mathcal{L}(X,\mathbb{R})} = 1$, $f(u) = 0$ for all $u \in U$, and $f(x_0) = \|x_0\|_X$.

b) Prove the Lax-Milgram Theorem as a corollary of the previous subproblem.

Remark: The generalization of Lax-Milgram to $V \neq W$ becomes indispensable, e.g. if you consider a system of PDEs in e.g. (u_1, u_2) and you want to allow u_2 to lie in a different space than u_1.
The FEM discretization procedure presented in the lecture can be applied seperately in the spaces V and W to obtain a discretization of such problems. This is called *mixed Finite Elements*.
There are equivalent formulations of the inf-sup condition, that are easier to verify than (2), depending on the problem. See e.g. Paragraph III.4 in the Braess' book (on the literature list).

See next sheet!

4. Let $D := (0,1)^2$, $1 > h > 0$, and define

$$f_h(x_1,x_2) := \begin{cases} 1 - \frac{x_1}{h} & x_1 \in [0,h], x_2 \in [0,1] \\ 0 & x_1 \in [h,1], x_2 \in [0,1] \ . \end{cases}$$

Show that for all $1 > h > 0$, f_h defines a class in $L^2(D)$ such that

$$\forall c > 0 \ \exists h \in (0,1) \ \text{ s. t. } \ \|f_h\|_{L^2(\partial D} > c\|f_h\|_{L^2(D)} \ .$$

This implies that, even for $u \in C^0(\bar{D})$, a "$L^2(D)$-trace inequality" of the form $\|u\|_{L^2(\partial D)} \leq c\|u\|_{L^2(D)}$ can not hold!

5. Let $D := (0,1)$, and $\mathcal{T} := \{x_i \ : \ x_i = \frac{i}{M}, i = 0,\ldots,M\}$. We recall (from the previous exercise sheet) the stiffness matrix for the bilinear form $a(u,v) := \int_a^b u'(x)v'(x)\,dx$ on $H_0^1(D)$, discretized on $\mathcal{T} \setminus \{x_0, x_M\}$ with piecewise linear FEM and the hat basis:

$$\mathbf{K} = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & -\frac{1}{h} & \frac{1}{h} \end{bmatrix} .$$

a) In Matlab, write a function `Cond_Mat_LFE1D.m` that computes for $h = 10^{-n}$, $n = 1,2,\ldots,100$ the condition number $\varkappa(\mathbf{K})$. What do you observe as $h \to 0$?
Hint: Do not assemble $\mathbf{K}$! Since you are given $\mathbf{K}$ a priori, you can spare the long time needed for assembly by `spdiags`. Moreover, a normal plot will not be very conclusive. Try `loglog` or `semilogx`.

b) For $h = 2^{-n}$, $n = 1,\ldots,13$, plot the condition numbers of the stiffness matrix and the mass matrix into the same plot. Can you explain this behaviour?

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Oct 7**, 2013