Exercise 3

1. In this task, we implement piecewise linear 2D Finite Elements.
A general remark: Unless indicated otherwise, vectors are always assumed to be columns, *except gradients*.

a) In 2D, we choose our reference element to be the standard 2-simplex $\hat{K} := \mathrm{convhull}([0,\ 0], [0,\ 1], [1,\ 0])$. Recall that the shape functions for piecewise linear FEM are

$$\begin{aligned}\hat{N}_0(\xi_1, \xi_2) &:= 1 - \xi_1 - \xi_2 \\ \hat{N}_1(\xi_1, \xi_2) &:= \xi_1 \\ \hat{N}_2(\xi_1, \xi_2) &:= \xi_2\ .\end{aligned}$$

Prove that the element stiffness matrix $\hat{\mathbf{A}}_i$ in element K_i is of the form

$$\hat{\mathbf{A}} = \mathrm{area}(K)\, \mathbf{N} J_i^{-1} J_i^{-\top} \mathbf{N}^T\ ,$$

where $\mathbf{N} := \begin{bmatrix} \nabla_{\boldsymbol{\xi}} \hat{N}_0 \\ \nabla_{\boldsymbol{\xi}} \hat{N}_1 \\ \nabla_{\boldsymbol{\xi}} \hat{N}_2 \end{bmatrix}$, and J_i is the Jacobi matrix of the affine element map

$$F_i : \hat{K} \to K_i\ .$$

b) Compute a similar formula for the element Mass matrix.

c) We use the assembly algorithm provided by LehrFEM. The file `assemMat_LFE.m` (provided) takes a function as input argument which computes the element matrices, and if necessary, the input of the element contribution-computation. See the LehrFEM manual.
Write functions `STIMA_Lapl_LFE.m` and `MASS_LFE.m` which compute for given 3 points (and α, in case of $\mathbf{M}$) the element contribution for $\mathbf{A}$ and $\mathbf{M}$. Use the formulae derived in the previous two subproblems. Look at the sparsity pattern of your assembled $\mathbf{A}$ and $\mathbf{M}$ with the command `spy`.

Hint: We provide `.p`-files for these functions. They contain reference solutions to test your code, but you cannot look at the code.

d) Write a function `assemLoad_LFE.m` that assembles the load vector. This works similar as in 1D, but the quadrature needs to be given in 2D. We provide the LehrFEM file `P706.m` that returns nodes and weights of a quadrature of order 7.

Please turn sheet!

e) Consider the domain $D := (0, 24) \times (0, 8)$ and on D, the Helmholtz problem

$$\Delta u(x, y) + k(x, y)^2 u(x, y) = 0, \quad u(x, y) = \begin{cases} \sin(\pi y/8) & \text{if } x = 0 \\ 0 & \text{if } y = 0, x = 24, y = 8 \end{cases}$$

where $k(x, y) := \begin{cases} 8 & \text{if } (x, y) \in (9, 15) \times (0, 4) \\ 2 & \text{elsewhere}. \end{cases}$ is piecewise constant.

We provide a mesh on D in the LehrFEM-structure as a `.mat`-file on the webpage. Write a matlab solver `Call_Helmholtz_LFE.m` that computes the approximate solution by pieewise linear FEM to this problem on the provided mesh. A skeleton can be downloaded from the website.

Hint: The function `get_BdEdges.m` extracts boundary edges from the Mesh. It is necessary to incorporate the boundary conditions.

2. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $\Gamma \subset \partial D$ some subset with positive measure, $|\Gamma| > 0$. Adjust the proof of the Second Poincaré inequality from the lecture to show that there exists a constant $c > 0$ such that

$$\|u\|_{L^2(D)} \leq c \|\nabla u\|_{L^2(D)}, \qquad \text{for all } u \in H^1_\Gamma(D) = \{u \in H^1(D) : u|_\Gamma = 0\}.$$

Where exactly is the condition $|\Gamma| > 0$ needed?

3. Let $u \in H^1(D)$ be some function, such that $\|\nabla u\|_{L^2(D)} = 0$. Prove that then u must be a constant function on D.

4. What is the dimension of the space $\mathcal{P}_p(K) \cap H^1_0(K)$ for a triangle $K \subset \mathbb{R}^2$ and $p \in \mathbb{N}$? For the simplest non-trivial case, describe a basis of this space.

5. For a triangle $K \subset \mathbb{R}^2$, find a basis of the space $\{\mathbf{u} \in (\mathcal{P}_1(K))^2 : \nabla \cdot \mathbf{u} = 0\}$.

6. This is an additional problem with increased difficulty.

Give a proof of the following version of Rellich's Theorem:

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain.
Then $H^m_0(D)$ is compactly embedded into $H^{m-1}_0(D)$.
In other words, every bounded sequence $(u_k)_k \subset H^m_0(D)$
contains a subsequence $(u_{k_j})_j$ which converges in the norm of $H^{m-1}(D)$.

For this it is sufficient (Why?) to prove that $H^1_0(D)$ is compactly embedded into $L^2(D)$.

See next sheet!

One possible approach, based on the Fourier transform is as follows:

1. Let a bounded sequence $(u_k)_k \subset H_0^1(D)$ be given. Argue that w.l.o.g. we may assume $u_k \in C_0^\infty(D) \subset C_0^\infty(\mathbb{R}^d)$.
2. Using Plancherel's identity, reformulate the assertion as one for the corresponding Fourier transforms.
3. Use the following fact (a special case of the (Banach-)Alaoglu Theorem):

   Let X be a reflexive Banach space.
   Then every bounded sequence contains a weakly convergent subsequence,

   to extract a suitable subsequence.
4. Show that
$$f \mapsto \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx$$
   defines a bounded linear functional on $L^2(D)$, and hence $\big((\mathcal{F}u_k)(\xi)\big)_k$ converges for every $\xi \in \mathbb{R}^d$.
5. Show
$$\int_{\{|\xi| \le R\}} \left|\mathcal{F}u_k(\xi) - \mathcal{F}u_j(\xi)\right|^2 d\xi \longrightarrow 0 \qquad (j, k \to 0)$$
   for every $R > 0$, using Lebesgue's dominated convergence theorem, and the boundedness of $\mathcal{F}f$ for every $f \in L^1(\mathbb{R}^d)$.
6. Argue $\|\xi_j \mathcal{F}u_k\|_{L^2(\mathbb{R}^d)} = \|\partial u_k\|_{L^2(\mathbb{R}^d)} \le \|u_k\|_{H^1(D)}$. Use this to finally prove
$$\int_{\{|\xi| > R\}} \left|\mathcal{F}u_k(\xi) - \mathcal{F}u_j(\xi)\right|^2 d\xi < \epsilon$$
   for sufficiently large $R = R(\epsilon)$.

Where exactly has the boundedness of the domain D been used?

Remark: Rellich's Theorem remains valid upon replacing $H_0^m(D)$ by the larger space $H^m(D)$, i.e. $H^m(D)$ is compactly embedded into $H^{m-1}(D)$, but the corresponding proof needs slightly more advanced tools.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Oct 14**, 2013