Exercise 4

1. Let $D \subseteq \mathbb{R}^2$ be an open, bounded domain. Instead of triangles, FEM can be defined on a *rectangular mesh*. To this end, we consider the *reference quadrilateral* $\hat{K} := (0,1)^2$ with vertices $\xi_0 := (0,0), \xi_1 := (1,0), \xi_2 := (1,1), \xi_3 := (0,1)$, and a regular partition of D into a finite set $\mathcal{T}$ of (straight-edged) rectangles[1].

a) As before, consider the reference space $\mathcal{S}^1(\hat{K}) := \{\varphi \in C^0(\hat{K}) \,:\, \varphi(\xi) = a + b\xi_1 + c\xi_2, a, b, c, \in \mathbb{R}\}$. Prove that there is no *nodal basis* $\{\hat{N}_i \,:\, i = 1, \dots, \dim\}$ of $\mathcal{S}^1(\hat{K})$, i.e. no basis that satisfies $\hat{N}_i(\xi_j) = \delta_{ij}$, $j = 0, \dots, 3$.

b) Let K be a quadrilateral with vertices $x_0, \dots, x_3$. Compute a formula for the bijective, *bilinear* element map $F_K : \hat{K} \to K$.

c) Compute a nodal basis $(\hat{N}_i)$ of the space of *bilinear* functions

$$Q^1(\hat{K}) := \{\varphi \in C^0(\hat{K}) \,:\, \varphi(\xi) = a + b\xi_1 + c\xi_2 + d\xi_1\xi_2, a, b, c, d \in \mathbb{R}\}.$$

Extend to a nodal basis $(b_i(x))_{i=1,\dots,M}$ of the bilinear FE-space

$$Q^1(D, \mathcal{T}) := \{\varphi \in H^1(D) \,:\, \varphi \circ F_K \in Q^1(\hat{K})\}\ .$$

d) As done for triangular FEM (see previous sheets), we define the (bilinear) local stiffness and mass matrices in element K with vertices $x_k, k = 0, \dots, 3$ by

$$(\mathbf{A}_K)_{kl} := \int_K \nabla\hat{N}_l(\xi)\cdot\nabla\hat{N}_k(\xi)\,dx, \quad (\mathbf{M}_K)_{kl} := \int_K \alpha(F_K(\xi))\hat{N}_l(\xi)\cdot\hat{N}_k(\xi)\,dx,$$

where $\alpha \in C^0(\bar{D})$ is given in advance. Compute their entries for constant α on K.

e) Write MATLAB functions `STIMA_BilFE.m` and `MASS_BilFE.m` that compute the contributions of a quadrilateral K given by a 4×2 array of vertices to the global stiffness and mass matrices. Assume that α is constant inside each element.

[1] a straight-edged rectangle with vertices $x_i, i = 0, \dots, 3$ is just the convex hull of the vertices. We always assume a quadrilateral to be straight-edged in this exercise.

Please turn sheet!

f) We provide you a file `assemMat_BilFE.m` which assembles global matrices analogously to `assemMat_LFE.m` from the previous sheet. Read through the code and try to understand each line. Things like `varargin`, `varargout` and the use of the `sparse` command as it is done in the code may be completely new to you if you did not do the programming exercise of sheet 3!

g) Write a function `assemLoad_BilFE.m` that assembles the global load vector with bilinear quadrilateral Finite Elements.

h) On the webpage, you will find a mesh file `quadMesh_Square.mat` containing data for a quadrilateral mesh on $D = (-1, 1)^2$. You will also find a function `refine_REG.m` that refines a given mesh regularly, just by "splitting" all elements (try to understand the code!).
Write a code `Call_Conv_BilFE.m` that computes the numerical solution of

$$-\Delta u(x) + = f(x)\ , \quad u|_{\partial D} \equiv u_D$$

and the H^1-norm of the error (using the uploaded function `H1Err_BilFE.m`) for several refinements of the given mesh, and compute the convergence rate for exact solutions

$$u_1(x) := \sin(2\pi x_1)\sin(\pi x_2), \text{ and } u_2(x) := (x_1 + 1)^{2/3}.$$

What difference do you observe, and how do you explain it?
Hint: If you want to work with a general Gauss-Legendre quadrature, you have to tensorize it. This means that given the 1D-quadrature rule with nodes $\{\eta_i\}$ and weights ω_i resulting from `gauleg.m`, take as nodes the product set $\{\xi_i\} \times \{\xi_i\}$, and as weights for the node (η_k, η_l), take $\omega_k\omega_l$. Save it in a struct `QuadRule` as you have done for triangular FEM.

2. Prove the following steps of the proof of the Bramble-Hilbert-Lemma in the lecture.

a) For all functions $v \in L^2(\widehat{K})$ and all multiindices $\underline{\alpha}$ with $|\alpha| = k$ it holds

$$\int_{\widehat{K}} \left(\frac{|\underline{\alpha}|}{\underline{\alpha}!} \int_0^1 s^{|\alpha|-1} \underline{\eta}^{\underline{\alpha}} v(s\underline{\eta}) ds \right)^2 d\underline{\eta} \le C \, \|v\|_0^2 \, ,$$

where $\widehat{K} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0, x_1 + x_2 < 1\}$ is the reference element in $\mathbb{R}^2$.

b) For all multiindices $\underline{\gamma}$ with $|\underline{\gamma}| = 1$ and all functions $v \in C^k(\hat{K})$ it holds

$$\partial^{\underline{\gamma}} \sum_{|\underline{\alpha}|=k} \frac{|\underline{\alpha}|}{\underline{\alpha}!} \int_0^1 s^{|\alpha|-1} \underline{\eta}^{\underline{\alpha}} \partial^{\underline{\alpha}} v(s\underline{\eta}) ds = \sum_{|\underline{\beta}|=k-1} \frac{1}{\underline{\beta}!} \underline{\eta}^{\underline{\beta}} \partial^{\underline{\beta}+\underline{\gamma}} v(\underline{\eta}) \, .$$

See next sheet!

3. Show that the function

$$u(x) = \log\log\frac{e}{|x|}\,, x \neq 0\,, \quad u(0) = 0\,,$$

belongs to the space $H^1_0(B)$, where $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball in $\mathbb{R}^2$.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Oct 21**, 2013