Exercise 5

1. Let $D := \{x \in \mathbb{R}^2 \ \ \|x\| < 1\}$. Any triangulation with straight edges will only approximate ∂D. To avoid this problem, we transform D with coordinates (x_1, x_2) to D with polar coordinates (r, ϑ).
Invariably, the change of coordinates will lead to a variational problem with non-constant coefficients of a particular form posed on spaces with non-standard boundary conditions.
Consider the boundary value problem

$$\begin{aligned} \Delta u = f & \quad \text{in } D, \\ u = 0 & \quad \text{on } \partial D \ . \end{aligned}$$

The transformation from polar to Cartesian coordinates (x_1, x_2) is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{\Phi}(r, \varphi) := r \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix},$$

and we have $D = \mathbf{\Phi}(D_p)$, $D_p := (0, 1) \times (0, 2\pi)$.

a) Compute the Jacobian matrix of $\mathbf{\Phi}(r, \vartheta)$, and verify

$$\nabla_{(r,\vartheta)} u(r, \vartheta) = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \vartheta}\mathbf{e}_\vartheta \ .$$

b) The variational formulation in polar coordinates with $u = u(r, \vartheta)$ is: find $u \in V$

$$\underbrace{\int_0^1 \int_0^{2\pi} \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} r \, dr d\vartheta}_{=:a(u,v)} + \underbrace{\int_0^1 \int_0^{2\pi} \frac{1}{r}\frac{\partial u}{\partial \vartheta}\frac{\partial v}{\partial \vartheta} \, dr d\vartheta}_{=:b(u,v)} = \underbrace{\int_0^1 \int_0^{2\pi} f \, v \, r \, dr d\vartheta}_{=:l(v)} \quad \forall v \in V,$$

where

$$V := \left\{ v \in H^1(D_p) \ : \ v(r = 1, \vartheta) = 0, \frac{\partial v}{\partial \vartheta}(r = 0, \vartheta) = 0, v(r, \vartheta = 0) = v(r, \vartheta = 2\pi) \right\}.$$

Explain why the additional conditions are necessary in the definition of V.

Please turn sheet!

c) We endow D_p with a quadrilateral tensor product mesh $\mathcal{M}$ defined by nodes (r_i, ϑ_j), $i, j = 0, \dots, n$, where $0 = r_0 < r_1 < \dots < r_n = 1$, and $0 = \vartheta_0 < \vartheta_1 < \dots < \vartheta_n = 2\pi$. We choose a discretization by bilinear nodal FEM: $Q^1_p(D, \mathcal{M}) := Q^1(D_p, \mathcal{M}) \cap V$. The space Q^1 on a quadrilateral mesh has been defined on Sheet 4.
Show that $\dim(Q^1_p(D, \mathcal{M})) = (n-1)^2 + n$.

d) Let $K_{ij} := (r_i, r_{i+1}) \times (\vartheta_j, \vartheta_{j+1})$ be a quadrilateral in $\mathcal{M}$. We define one shape function by

$$b^1_K(r, \vartheta) := \frac{r - r_i}{r_i - r_{i+1}} \frac{\vartheta - \vartheta_j}{\vartheta_j - \vartheta_{j+1}}.$$

Compute formulae for the other shape functions b^2_K, b^3_K, b^4_K.

e) Write a matlab function `STIMA_LaplPolar_a.m` that computes the stiffness matrix entries induced by the bilinear form $a(u, v)$.

f) We need to implement the boundary condition that u must be constant. To this end, degrees of freedom at the left-hand edge have to be replace by only one d.o.f. corresponding to the constant. Namely, we replace the two basis functions b^1_K, b^4_K defined at the boundary by only one $b^0_K := b^1_K + b^2_K$. Consider an element $K := K_{0,j}$ at the left-hand edge of D_p.
A code `STIMA_LaplPolar_b.m` that computes the stiffness matrix induced by the bilinear form $b(u, v)$ on D_p can be downloaded from the page. It does not distinguish elements at the left-hand edge from the others. Modify it, such that for $i = 0$, it uses the new defined shape function b^0_K.
Hint: All this means that there is only one basis function associated with all vertices at $r = 0$. Its restriction to each element abutting $r = 0$ is the sum of the local shape functions associated with vertices located on the $r = 0$ side of D_p.

g) Write a function `assemMatPolar_BFE.m` that assembles the stiffness matrices for $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ on an *equidistant* mesh for given n.

h) On the webpage, you will find a function `assemLoadPolar_BFE.m` that assembles the load vector for $f \equiv 1$. Write a function that solves the Dirichlet Poisson problem (1.) on the disc D using the transformed problem as described in the previous subproblems, with $f \equiv 1$.

i) For $\alpha \in (0, 2\pi)$ a fixed angle, let $D_\alpha := \{(r, \vartheta) \in D : \vartheta < \alpha\}$ be thedisc segment with interiour opening angle α. Modify your `Call` file such that it solves the problem on D_α instead of D. Test the code for $\alpha = \frac{2}{3}$. What can you say about the regularity of the solution?

See next sheet!

2. Let $\mathcal{Q}$ be a tiling of the domain D into quadrilaterals. Prove that the space

$$S^p(D,\mathcal{Q}) = \{u \in C^0(D) : u|_K \circ F_K \in Q_p \,, \forall K \in \mathcal{Q}\}\,,$$

where F_K maps $\widehat{K} = (-1,1)^2$ bijectively onto K, and $Q_p = \text{span}\,\{\xi_1^i \xi_2^j\}_{0\le i,j\le p}$.

Prove that $S^p(D,\mathcal{Q})$ is conforming, i.e. $S^p(D,\mathcal{Q}) \subset H^1(D)$.

3. Consider a uniform tiling $\mathcal{Q} = \{Q_{i,j}\}_{i,j=0}^{N-1}$ of the unit square $D = (0,1)^2$ into squares $Q_{i,j} = (\frac{i}{N}, \frac{i+1}{N}) \times (\frac{j}{N}, \frac{j+1}{N})$. Moreover, we consider the spaces

$$V_N = S^1(D,\mathcal{Q}) = \{u \in C^0(D) : u|_{Q_{i,j}}(x,y) = a + bx + cy + dxy\}\,,$$

i.e. the space of piecewise bilinear functions w.r.t. $\mathcal{Q}$.

Estimate $e_N(u)_j = \inf_{v\in V_N} \|u - v\|_j$, $j = 0,1$, for the functions

a) $u_1(x,y) = \sin(\pi x)\sin(\pi y)$,

b) $u_2(x,y) = (x+y)^2$.

Express the error in terms of N, and in terms of the number N_{dof}, the number of degrees of freedom used for the respective approximation.

4. Let D_0 be a compact subset of the bounded domain D. Then we define

$$\delta_i^h u(x) = \frac{u(\underline{x} + h\underline{e}_i) - u(\underline{x})}{h}\,, \qquad i = 1,\dots,d\,,$$

the ith difference quotient, where x in D_0 and $h \in \mathbb{R}$ so that $0 < |h| < \text{dist}(D_0, \partial D)$. Moreover, we put $\delta^h u = (\delta_1^h u, \dots, \delta_d^h u)$.

Prove the following relations between difference quotients and derivatives.

a) Let $u \in H^1(D)$. Then for every compact subset $D_0 \subset D$

$$\|\delta^h u\|_{L^2(D_0)} \le C\,\|\underline{\nabla} u\|_{L^2(D)}\,, \qquad 0 < |h| < \tfrac{1}{2}\,\text{dist}(D_0, \partial D)\,,$$

for some constant C independent of h and u.

b) Let $u \in L^2(D)$, and assume that there exists a constant C such that

$$\|\delta^h u\|_{L^2(D_0)} \le C$$

for all $0 < |h| < \frac{1}{2}\,\text{dist}(D_0, \partial D)$. Then

$$u \in H^1(D_0) \quad \text{with} \quad \|\underline{\nabla} u\|_{L^2(D_0)} \le C\,.$$

Please turn sheet!

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Oct 28**, 2013