Exercise 6

1. Consider $-u'' + u = f$ on $D = (0,1)$ with $u(0) = u(1) = 0$. Take $u(x) = x^\alpha(1-x)$ as the exact solution.

a) For exactly which $\alpha \in \mathbb{R}$ is $u \in H_0^1(D)$?

b) Consider a quasi-uniform family of meshes $\mathcal{T}_h$ on D, $0 < h \leq 1$. Let $N = N(h)$ denote the dimension of the space $V_N := S^1(D, \mathcal{T}_h) \cap H_0^1(D)$. Let $u_N \in V_N$ denote the Galerkin approximation of u in V_N. For $1/2 < \alpha < 3/2$ and for $\alpha > 3/2$, what is the expected convergence rate for $\|u - u_N\|_{H^1(D)}$ as $h \searrow 0$? Express the rate in terms of the mesh width h, as well as in terms of the number of degrees of freedom N.

c) Implement and verify numerically.

d) Now use quadratic elements in your implementation. What do you observe?

e) Let $1/2 < \alpha < 3/2$. For a budget of N degrees of freedom, derive a distribution of nodes of $\mathcal{T}_h$ such that the $H^1(D)$ error is $\mathcal{O}(N^{-1})$ as $N \nearrow \infty$.

Without proof, you may use the following statement about the error of the piecewise linear nodal interpolant:

Let $h(x) \in S^1(D, \mathcal{T})$ be such that

$$h(x_j) = h_j + h_{j+1}, \quad j = 1, \dots, N, h_{N+1} = 0\,,$$

where $h_j = x_j - x_{j-1}$. Then $h(x) \geq h_j$ for all $x \in (x_{j-1}, x_j)$ and

$$\|v - \Pi_{\mathcal{T}}^1 v\|_{H^1(D)}^2 \leq \frac{1}{\pi^2} \|hv''\|_{L^2(D)}^2\,.$$

2. Let $\alpha \in (0, 2\pi)$, and put

$$D_\alpha = \{x \in \mathbb{R}^2 : (x_1, x_2) \cong (r, \phi) \text{ with } 0 < r < 1, 0 < \phi < \alpha\}$$

a sector with inner angle α.

a) Solve the Laplace equation $\Delta u(r, \phi) = 0$ on D_α with boundary conditions $u(r, 0) = u(r, \alpha) = 0$ for all $r \in (0,1)$ by a separation-of-variables approach.

Please turn sheet!

b) Classify for all your solutions their maximal Sobolev regularity in dependence of α, i.e. for every solution determine the maximal integer m such that it belongs to $H^m(D_\alpha)$.

3. As you (should) have seen in the previous exercise, for non-convex polygons or sectors, solutions of the Laplace equation not necessarily are square integrable. Similarly, solutions to Poisson's equation $-\Delta u = f$ with homogeneous Dirichlet boundary conditions need not belong to $H^{m+2}(D)$ for $f \in H^m(D)$ for such domains. However, there is a certain remedy to this failure: One needs to add weights in the definition of the respective norms.

To get an idea for this, we consider spaces $H^m_\gamma(D_\alpha)$, where D_α is a sector as above. These are defined as the completion of $C_0^\infty(D_\alpha)$ with respect to the norm

$$\|u\|^2_{H^m_\gamma(D_\alpha)} = \sum_{|\underline{\beta}|\leq m} \int_{D_\alpha} \left| |\underline{x}|^\gamma D^{\underline{\beta}} u(\underline{x}) \right|^2 d\underline{x}\,, \qquad \gamma \in \mathbb{R}\,.$$

a) Consider the functions $u_\lambda(\underline{x}) = |\underline{x}|^\lambda$. For which values of $\lambda \in \mathbb{R}$ do they belong to $H^2_\gamma(D_\alpha)$?

b) Once more, consider the solutions of $\Delta u = 0$ on D_α with $u(r, 0) = u(r, \alpha) = 0$. Determine weight exponents γ such that these solutions belong to $H^2_\gamma(D_\alpha)$.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Nov 4**, 2013