Exercise 7

1. (Duality techniques I) We consider an annulus $D := \{x \in \mathbb{R}^2 \ : \ \|x\|_2 \in (r, R)\}$ with inner and outer radii $0 < r < R < \infty$. We denote by $\Gamma_0 := rS^1$, $\Gamma_1 := RS^1$ the inner and outer connected component of ∂D (Here, S^1 is the unit sphere in $\mathbb{R}^2$).
This can be viewed as the section of a water pipe of thickness $R - r$ conduction e.g. water in the inner part. Given outer and inner temperatures u_0, $u_1 \in \mathbb{R}$, the stationary temperature distribution $u(x)$ is implicitly given by

$$-\Delta u = 0 \text{ on } D, \qquad u|_{\Gamma_i} \equiv u_i \ . \tag{1}$$

Instead of u, we are actually interested in the *heat flux* through Γ_1 and Γ_2, which are linear output functionals defined by

$$F_i : H^1(D) \to \mathbb{R} \quad v \mapsto F_i(v) := \int_{\Gamma_i} \nabla u(x) \cdot \vec{\boldsymbol{\nu}}(x) \, dx \ ,$$

where $\vec{\boldsymbol{\nu}}$ is the outward pointing normal vector.

a) Write a matlab function `distAnnulus.m` that takes $x \in \mathbb{R}^2$ as an input and returns the *signed distance to* ∂D, defined by $\begin{cases} \operatorname{dist}(x, \partial D) & \text{if } x \in \bar{D} \\ -\operatorname{dist}(x, \partial D) & \text{if } x \in \mathbb{R} \setminus \bar{D} \ . \end{cases}$
With this function, use `distmesh` or the provided LehrFEM function `init_Mesh.m` to construct a mesh on D.

b) What is the big weakness of this mesh? What would be the easiest way to get rid of this issue?

c) Write a function `evalFluxFun.m` that evaluates the flux functionals F_i for a given mesh on D and a given numerical solution u_h on this mesh.

d) Write a function `Call_FluxConv.m` that evaluates $F_i(u_h)$ for different meshes on D, and computes the error $|F_i(u_h) - F_i(u^*)|$, where u^* is the solution obtained on a very fine mesh. To refine initial meshes, use the function `refine_REG.m` used in previous exercise sheets. Use $u_0 := 10, u_1 := 60$ What is the convergence rate you would expect from the duality theory? What convergence rate would you expect?
Hint: Use the functions you got from anteriour exercises. They are contained in the reference solutions.

Please turn sheet!

e) Prove that F_i is not a bounded functional on $H^1(D)$.
Hint: Define a function $v(r)$ which lies in $H^1(D)$, but for which "$F_1(v) = \infty$".

f) We define a cut-off $\psi \in C^0(\bar{D}) \cap H^1(D)$, such that $\psi|_{\Gamma_0} \equiv 1$ and $\psi|_{\Gamma_1} \equiv 0$ and $\psi(x) \in [0,1]$ for all $x \in D$. Let $F_0^*(v) := \int_D \nabla u \cdot \nabla \psi \, dx$.
Prove that F_0^* is a bounded linear functional on $H^1(D)$ and that $F_0^*(u) = F_0(u)$ for all solutions u to (1) (and all $u_0, u_1 \in \mathbb{R}$).

g) Define a bounded linear functional $F_1^*(D)$ on $H^1(D)$ such that $F_1^*(u) = F_1(u)$ for all solutions u to (1).

h) Change your function `evalFluxFun.m` such that it computes $F_i^*(u)$ instead of $F_i(u)$. Re-run `Call_FluxConv.m`. What did change, and why?

2. **(Duality techniques II)** Let $D \subseteq \mathbb{R}^2$ be an open, bounded domain, $a(\cdot,\cdot)$ be a continuous, coercive bilinear form on $H_0^1(D)$, $l \in H_0^1(D)'$ be a continuous linear functional on $H_0^1(D)$ and u be the solution to

$$\text{Find } u \in H_0^1(D) \text{ such that } a(u,v) = l(v) \ \forall v \in H_0^1(D)\ . \tag{2}$$

Let $\{V_N\}_{N\in\mathbb{N}}$ be a family of *conforming nested Lagrangian F.E. spaces*, i.e. $V_N = S^p(D, \mathcal{T}_N) \cap H_0^1(D)$ for some shape-regular triangulation $\mathcal{T}$ such that $V_N \subseteq V_M$ whenever $M > N$, some fixed $p \in \mathbb{N}$ and where $N = \dim(V_N)$. Let u_N be the Galerkin approximation on V_N, i.e. $a(u_N, v) = l(v)$ for all $v \in V_N$.
We have seen in the lecture that $\|u - u_N\|_{H^1(D)} = O(N^{-\frac{\min(p+1,s)-1}{2}})$ as $N \to \infty$. Your task is to compute the convergence rate *in the* $L^2(D)$*-Norm*, i.e. to find $\xi > 0$ s.t. $\|u - u_N\|_{L^2(D)} = O(N^{-\frac{\xi}{2}})$, as $N \to \infty$.

a) Define a linear, continuous functional F on $H_0^1(D)$, such that $F(u) - F(u_N) = \|u - u_N\|_{L^2(D)}^2$.

b) Consider the dual problem

$$\text{Find } f_F \in H_0^1(D) \text{ such that } a(f_F, v) = F(v) \ \forall v \in H_0^1(D)\ . \tag{3}$$

What conditions on D must be satisfied s.t. $u \in H^2(D)$ and $\exists c > 0$ independent of F, u, N, s.t. $|f_F|_{H^2(D)} \leq c\|u - u_N\|_{L^2(D)}$?

c) Under the assumption that these conditions are satisfied, find the $L^2(D)$ convergence rate, i.e. the $\xi > 0$ from above.
Hint: Prove that $\inf_{v\in V_N} \|f_F - v\|_{H^1(D)} \leq cN^{-\frac{1}{2}}\|u - u_N\|_{L^2(D)}$ and conclude with the duality estimate from the lecture.

See next sheet!

d) Write a matlab code `Call_ConvL2_LFE.m` which computes the L^2-convergence rates for the problem $-\Delta u = 0$, $u|_{\partial D} \equiv g$. To compute the L^2-errors, download the function `L2Err_LFE.m`. Use the L-shaped domain $D := (-1,1) \times [0,1) \cup (-1,0) \times (-1,0)$ and the coarse initial mesh given as a `.mat` file on the website. First use as exact solution $u(x) = \sin(\pi x_1)\sin(\pi x_2)$ where the regularity assumption from problem c) is satisfied. Next, use $u(x) = u(r,\varphi) := r^\lambda \sin(\lambda\varphi)$, where $\lambda = \frac{2\pi}{3}$ and where (r,φ) are polar coordinates centred at the origin. Compare the experimentally obtained L^2-rate to the expected H^1-rate.

3. (Variational crimes) Let D be an open, bounded domain, $V \subseteq H^1(D)$ closed subspace, and consider the variational problem to find $u \in V$ s.t. $a(u,v) = l(v)$ for all $v \in V$, where $l \in V'$, i.e. l is a continuous linear functional on V, and a is a continuous, coercive bilinear form on V.
Let $\{V_N\}_{N\in\mathbb{N}}$ be a family of subspaces of V with $\dim(V_N) = N$. Our task is to find $u_N \in V_N$, s.t. $a_N(u_N, v) = l_N(v)$ for all $v \in V_N$ and all $N \in \mathbb{N}$, where $l_N \in (V_N)'$, and $\{a_N\}$ is a family of continuous bilinear forms on V_N which is *uniformly coercive*, i.e. $\exists \alpha > 0$, s.t. $\forall N \in \mathbb{N}$, $v \in V_N$, $a_N(v,v) \geq \alpha \|v\|^2_{H^1(D)}$. Prove that there is a constant $c > 0$ independent of N, s.t.

$$\|u - u_N\|_{H^1(D)} \leq \frac{c}{\alpha}\left(\inf_{v_N \in V_N\setminus\{0\}} \left\{ \|u - v_N\|_{H^1(D)} + \sup_{w_N \in V_N} \frac{|a(v_N,w_N) - a_N(v_N,w_N)|}{\|w_N\|_{H^1(D)}} \right\} + \sup_{w_N \in V_N\setminus\{0\}} \left\{ \frac{|l(w_N) - l_N(w_N)|}{\|w_N\|_{H^1(D)}} \right\} \right) \tag{4}$$

Hint: Let $v_N \in V_N$. Observe $\|u - u_N\| \leq \|u - v_N\| + \|u_N - v_N\|$. Now, compute an inequality $\alpha\|u_N - v_N\|^2 \leq \cdots \leq$[similar to what is on the right hand side, but no fractions nor inf/sup]. From there, it is easy to conclude the claim.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Nov 11**, 2013