Exercise 8

1. On a domain D with a mesh family $\{\mathcal{T}_h\}_{h>0}$, write a piecewise linear, nodal FEM-solver for the semilinear problem

$$-\Delta u(x) + u(x)^3 = f(x), \quad u|_{\partial D} \equiv g,$$

using LehrFEM[1] functions. Proceed the following way:

a) Let K be an element in an arbitrary triangulation $\mathcal{T}$. Let $\{N_i,\ i = 0,1,2\}$ be the linear local shape functions on K. Compute the entries of the matrix $(\mathbf{A}''_K)_{kl} := \int_K N_k(x)^3 N_l(x)\, dx$.

b) Recall that the bilinear form in the variational formulation for the PDE considered here can be split up $a(u,v) := \int_D \nabla v \cdot \nabla u \, dx + \int_D u^3 v \, dx$. Hence, the corresponding Galerkin matrix splits up into $\mathbf{A} = \mathbf{A}' + \mathbf{A}''$, where $\mathbf{A}'$ is the usual stiffness matrix and $\mathbf{A}''$ is the global matrix of $\mathbf{A}''_K$ computed in the previous subproblem. This motivates the following exercise:

Write an matlab function `A2=Cubic_LFE(Vertices, flag)` which takes a list of three vertices and an element flag as an input, and returns the local contribution to the matrix $\mathbf{A}''$ computed in the previous subproblem.

Hint 1: Although the flag will not be used, it is needed to be compatible with the LehrFEM assembly.

Hint 2: Copy the content of `MASS_LFE.m` and change function name and content to obtain the desired function.

c) Let D be the L-shaped domain $D := (-1,1)^2 \setminus ([0,1) \times (-1,0])$. Generate an initial mesh of meshsize $h_0 := 0.25$ on D using the LehrFEM function `init_Mesh.m`.

d) Write a script `Call_CubicLapl_LFE.m` that computes the linear FEM solution on this initial mesh and 4 regular refinements of it and computes the L^2 and H^1 errors. As exact solutions, use $u_1(x) := \sin(2\pi x_1)\cos(2\pi x_2)$ and $u_2(x) = u_2(r,\varphi) = r^{\frac{2}{3}} \sin(\frac{2}{3}\varphi)$, with polar coordinates (r,φ) centered at the reentering corner $(0,0)$. Find the algebraic convergence rates.

[1] A collection of the most important LehrFEM files can be downloaded from the lecture's homepage. See the handout on the webpage for more information.

Please turn sheet!

Hint: To refine the initial mesh constructed in subproblem c), use `refine_REG.m`. To construct the second part of the Galerkin matrix, use the assembly master file `assemMat_LFE.m` with the function `Cubic_LFE.m` from subproblem b)

2. Regularity results for semilinear equations.

a) Let $D \subset \mathbb{R}^2$ be an arbitrary bounded Lipschitz domain. Prove that whenever $u \in H^m(D)$ it follows $u^3 \in H^{m-1}(D)$ for all $m \in \mathbb{N}$.

b) Now consider the problem

$$-\Delta u + u^3 = f \quad \text{on } D\,, \qquad u|_{\partial D} = 0\,.$$

Existence and uniqueness of a solution $u \in H_0^1(D)$ for every $f \in H^{-1}(D)$ was shown in the lecture. Moreover, for D either a convex polygon or sufficiently smooth domain we have seen $u \in H^2(D)$ in case $f \in L^2(D)$. This shall be generalized.

Prove: If $f \in H^m(D)$ for some $m \in \mathbb{N}$ and D is a sufficiently smooth domain, then it follows $u \in H^{m+2}(D)$.

3. Eigenvalues vs. Poincaré's inequality

a) Denote by $\phi_j \in H_0^1(D)$ the eigenfunctions of the Dirichlet-Laplacian with eigenvalue λ_j, i.e. the solutions of the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{on } D\,, \qquad u|_{\partial D} = 0\,.$$

It is known that there are countably many, pairwise H^1-orthogonal such eigenfunctions $(\phi_j)_{j\in\mathbb{N}}$, and the corresponding eigenvalues shall be ordered in magnitude, i.e. $0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Give a proof of Poincaré's inequality $\|u\|_0 \leq C_D \|\nabla u\|_0$ for $u \in H_0^1(D)$ based on the fact that the system $(\phi_j)_{j\in\mathbb{N}}$ is a basis both in $L^2(D)$ and $H_0^1(D)$.

b) Derive an expression for the optimal constant in Poincaré's inequality. Does a limiting function exist, i.e. a function $u_0 \in H_0^1(D)$, for which the inequality with optimal constant turns into an equation?

c) In the case $D = (0, 1)$, determine the exact value of Poincaré's constant, i.e. the optimal constant in Poincaré's inequality.

d) What changes in the case of Neumann boundary conditions, respectively the Second Poincaré inequality?

See next sheet!

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Nov 18**, 2013