Exercise 9

Legendre polynomials can be introduced in a number of ways: They are the solutions of Legendre's differential equation

$$\frac{d}{dt}\Big((1-t^2)\frac{d}{dt}L_n(t)\Big) + n(n+1)L_n(t) = 0\,,$$

can be obtained by Gram-Schmidt orthogonalization of the monomial system $\{1, x, x^2, \ldots\}$ in the Hilbert space $L^2([-1,1])$, or more directly either by Rodriguez' formula

$$L_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n\,, \qquad n \geq 0\,,$$

or via the recursion

$$(n+1)L_{n+1}(t) = (2n+1)tL_n(t) - nL_{n-1}(t)\,, \qquad L_0(t) = 1\,, \quad L_1(t) = t\,.$$

Together with Chebyshev polynomials they are particularly useful for the implementation and analysis of certain Finite element methods, which shall be briefly introduced in this Exercise sheet.

1. 1D-hierarchic shape functions for $\mathcal{S}_p^0$

On the interval $[-1,1]$, we define the following local shape functions $\{\widehat{b}_n\}_{n=1}^p$:

$$\begin{cases} \widehat{b}_1(\xi) \;=\; \frac{1-\xi}{2}, \quad \widehat{b}_2(\xi) \;=\; \frac{1+\xi}{2}, \\ \widehat{b}_n(\xi) \;=\; \sqrt{\dfrac{2n-3}{2}}\displaystyle\int_{-1}^{\xi} L_{n-2}(t)\,dt, \quad \forall n \geqslant 3\,. \end{cases} \tag{1}$$

a) Prove that the mass matrix

$$\widehat{M} \;=\; \Big(\int_{-1}^{1}\widehat{b}_i(\xi)\,\widehat{b}_j(\xi)\,d\xi\Big)_{i,j=1}^{p}$$

Please turn sheet!

is given by:

$$\widehat{M} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{10}} & 0 & \cdots & \cdots & \cdots & 0 \\ & \frac{2}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{10}} & 0 & \cdots & \cdots & \cdots & 0 \\ & & \widehat{M}_{2,2} & 0 & \widehat{M}_{2,4} & \ddots & & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & & \widehat{M}_{i,i} & 0 & \widehat{M}_{i,i+2} & \ddots & \vdots \\ & & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & & \widehat{M}_{p-2,p-2} & 0 & \widehat{M}_{p-2,p} \\ & & & & & & & \widehat{M}_{p-1,p-1} & 0 \\ \text{sym} & & & & & & & & \widehat{M}_{p,p} \end{pmatrix}$$

with

$$\widehat{M}_{i,i} = \frac{2}{(2i-1)(2i-5)} \quad \text{and} \quad \widehat{M}_{i,i+2}\frac{-1}{(2i-1)\sqrt{(2i-3)(2i+1)}}$$

Hint: Prove that

$$\begin{cases} \widehat{b}_1(x) = \dfrac{L_0(x)-L_1(x)}{2}, \quad \widehat{b}_2(x) = \dfrac{L_0(x)+L_1(x)}{2}, \\ \widehat{b}_n(x) = \dfrac{1}{\sqrt{2(2n-3)}}\big(L_{n-1}(x)-L_{n-3}(x)\big), \end{cases}$$

and use the orthogonality relation

$$\int_{-1}^{1} L_n(x)\, L_m(x)\, dx = \begin{cases} \dfrac{2}{2n+1} & \text{for } n=m, \\ 0 & \text{otherwise.} \end{cases}$$

and the formula

$$L_n(x) = \frac{L'_{n+1}(x)-L'_{n-1}(x)}{2n+1}.$$

b) Show that the stiffness matrix

$$\widehat{K} = \Big(\int_{-1}^{1} \widehat{b}'_i(\xi)\, \widehat{b}'_j(\xi)\, d\xi\Big)_{i,j=1}^{p}$$

is given by:

$$\widehat{K} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\ & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ & & 1 & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 \\ \text{sym} & & & & & 1 \end{pmatrix}$$

See next sheet!

c) On the interval $(0, 1)$, we introduce the following mesh:

$$\mathcal{T} \;=\; \{K_i\}_{i=1}^{N}$$

with

$$K_i = \big((i-1)h; ih\big),\;\; N \in \mathbb{N} \text{ and } h = \frac{1}{N}.$$

Moreover, we define the element maps $\varphi_i : K_i \to (-1, 1)$

$$\varphi_i(x) \;=\; -1 + 2 \cdot \frac{x - (i-1)h}{h}.$$

We then define the following set of functions $\{b^i\}_{i=1}^{Np+1}$:

- For $i = 0, \ldots, N$

$$b^{ip+1}(x) = \begin{cases} \widehat{b}_2(\varphi_i(x))\,, & \text{if } x \in K_i\,, \\ \widehat{b}_1(\varphi_{i+1}(x))\,, & \text{if } x \in K_{i+1}\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

  where $K_0 = K_{N+1} = \emptyset$.
- For $i = 1, \ldots, N$ and $j = 2, \ldots, p$

$$b^{(i-1)p+j}(x) = \begin{cases} \widehat{b}_{j+1}(\varphi_i(x))\,, & \text{if } x \in K_i\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Show that $\{b^i\}_{i=1}^{Np+1}$ is a basis of

$$\mathcal{S}_p^0(\mathcal{T}) \;=\; \big\{u \in C^0([0,1]) \;:\; u|_{K_i} \in \mathcal{P}_p(K_i)\big\}$$

d) Compute the mass matrix

$$M_N \;=\; \Big(\int_0^1 b^i(x)\, b^j(x)\; dx\Big)_{i,j=1}^{Np+1}$$

and the stiffness matrix

$$K_N \;=\; \Big(\int_0^1 (b^i)'(x)\; (b^j)'(x)\; dx\Big)_{i,j=1}^{Np+1}$$

in terms of h, $\widehat{M}$ and $\widehat{K}$.

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2. In order to gain higher accuracy, instead of refining the width h of the elements, one can enlarge the polynomial degree. The goal of this exercise is to obtain an experimental result on the convergence rate of this strategy (which is by the way called p-FEM, as opposed to h-FEM).
We treat a simple 1D case with a uniform polynomial degree $p \in \mathbb{N}$ which is to be increased. On $D := (0, 1)$, consider the equation

$$-u''(x) = f(x), \quad u(0) = u(1) = 0 \, .$$

a) Our basis functions are chosen as $\widehat{b}_n$, $n = 1, \dots, p$ in (1), of the previous exercise. Using the matlab command `spdiags`, write a function
`Aloc=STIMA_Lapl_pFE1D(Vertices, flag, p)`
which computes the local contribution A_k to the global stiffness matrix A from element k (possibly marked with a flag `flag`) bounded by the vertices passed as input in the list `Vertices`. Use the formulae from the previous exercise.

b) Write a function `A=assemMat_pFE1D(Mesh, EHandle, p)` which assembles a global matrix A by the function `EHandle` which returns the local contributions. Test it using `STIMA_Lapl_pFE1D` from the previous subproblem.

Hint: Add minor changes in `assemMat_LFE.m`

c) Change the LehrFEM functions `assemLoad_LFE.m` and `assemDir_LFE.m` to functions `assemLoad_pFE1D.m` and `assemDir_pFE1D.m` respectively, such that they assemble the load vector and commit boundary term manipulations with p-FEM. The polynomial degree has to be passed as input.

d) Let $D := (0, 1)$, $u_1(x) := \sin(4\pi x)$, $u_2(x) := |x - \frac{1}{2}|^{\frac{3}{4}}$. Construct an initial mesh with $N = 20$ points using `linspace`. Solve the equation on this mesh with local polynomial degrees $p = 1, 4, 8, 16, 24$ with u_1 and u_2 as exact solutions and compute the errors in the H^1 and L^2 norms. In order to estimate the convergence rate, draw a `semilogy` for the error to u_1 and a `loglog` plot for the error to u_2. What do you observe, and how would you explain it?

3. Inverse estimates

a) Let T be a nondegenerate κ-shape-regular simplex in $\mathbb{R}^d$ for $d \geq 1$ (with T being an interval in the case $d = 1$). Let $\mathcal{P}^p(T)$ denote the polynomials on T of degree at most $p \in \mathbb{N}$. Show that there exists $C_{\kappa,p} > 0$ (independent of T) such that

$$\|D^k u\|_{L^2(T)} \leq C_{\kappa,p} h_T^{-k} \|u\|_{L^2(T)} \quad \forall u \in \mathcal{P}^p(T)$$

where h_T denotes the diameter of T.

For a κ-shape-regular triangulation $\mathcal{T}$ of a domain $D \subset \mathbb{R}^d$ prove

$$\|D^k u\|_{L^2(D)} \leq C_{\kappa,p} h_{\mathcal{T}}^{-k} \|u\|_{L^2(D)} \quad \forall u \in S^{p,1}(D, \mathcal{T}).$$

Which combinations of k and p are allowed?

See next sheet!

b) Consider now the reference element $\widehat{K} = (-1, 1)$. Show that for each $k \in \mathbb{N}$ there exists $\widehat{C}_k > 0$ such that for any $p \in \mathbb{N}$

$$\|u^{(k)}\|_{L^2(\widehat{K})} \leq \widehat{C}_k p^{2k} \|u\|_{L^2(\widehat{K})}, \quad \forall u \in \mathcal{P}^p(\widehat{K}).$$

Generalize to $K = (a, b) \subset \mathbb{R}$.

Hint: Use expansions into Legendre polynomials. In particular, first express L_n' in terms of Legendre polynomials L_k, $k \leq n - 1$.

c) Compare the estimates from **a)** for $d = 1$ and **b)** in terms of the number of degrees of freedom.

Please justify your answers and give precise references to your sources (lecture notes, text books, exercise class, etc.). Consult the lecture homepage (`www.math.ethz.ch`) for administrative and further questions.

Due date: **Mon**, **Nov 25**, 2013