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Solutions 2

1. (i) For any set $X \subseteq \Omega$, we have

$$|P_1(X) - P_2(X)| = \left|\sum_{\omega \in X} P_1(\omega) - P_2(\omega)\right| \leq \sum_{\omega \in X} |P_1(\omega) - P_2(\omega)| \quad \text{and}$$
$$|P_1(X^C) - P_2(X^C)| = \left|\sum_{\omega \notin X} P_1(\omega) - P_2(\omega)\right| \leq \sum_{\omega \notin X} |P_1(\omega) - P_2(\omega)|,$$

where $X^C := \Omega \setminus X$ denotes the complement of X. These two inequalities, together with $P_1(X^C) = 1 - P_1(X)$ and $P_2(X^C) = 1 - P_2(X)$, yield

$$2|P_1(X) - P_2(X)| \leq \sum_{\omega \in \Omega} |P_1(\omega) - P_2(\omega)|,$$

but since the choice of X was arbitrary, we have

$$\delta(P_1, P_2) \leq \frac{1}{2} \sum_{\omega \in \Omega} |P_1(\omega) - P_2(\omega)| = \frac{1}{2} ||P_1 - P_2||_1.$$

Conversely, let $Y := \{\omega \in \Omega : P_1(\omega) \geq P_2(\omega)\}$. We have

$$\sum_{\omega \in Y} |P_1(\omega) - P_2(\omega)| = \sum_{\omega \in Y} P_1(\omega) - P_2(\omega) = P_1(Y) - P_2(Y) \quad \text{and}$$
$$\sum_{\omega \notin Y} |P_1(\omega) - P_2(\omega)| = \sum_{\omega \notin Y} P_2(\omega) - P_1(\omega) = P_2(Y^C) - P_1(Y^C).$$

By adding these two equalities, we obtain

$$P_1(Y) - P_2(Y) = \frac{1}{2} \sum_{\omega \in \Omega} |P_1(\omega) - P_2(\omega)|,$$

hence $\delta(P_1, P_2) \geq \frac{1}{2} ||P_1 - P_2||_1$, finishing the proof.

(ii) The ε-robustness is related to the l_∞ norm, as P on $\{0, 1\}^n$ is ε-robust if

$$||P - \mathbb{U}_n||_\infty < \varepsilon.$$

2. We will prove a stronger version of this claim: for any distribution P on $\{0,1\}^n$,

$$\delta(\mathcal{P}_{f,P}, \mathbb{U}_m) \geq \frac{1}{2}.$$

Let $Y = \text{Im}(f) \subseteq \{0,1\}^m$ be the image of f. Clearly $|Y| \leq 2^n \leq 2^{m-1} = \frac{1}{2} \cdot 2^m$. Let $X \subseteq \{0,1\}^m$ be any set of size $|X| = \frac{1}{2} \cdot 2^m$ disjoint from Y. We have $\mathbb{P}_{x \sim \mathbb{U}_m}[x \in X] = \frac{1}{2}$. But since X is disjoint from the image of f, we must have

$$\mathbb{P}_{x \sim \mathcal{P}_{f,P}}[x \in X] = 0,$$

which implies that $\delta(\mathcal{P}_{f,P}, \mathbb{U}_m) \geq \frac{1}{2}$, concluding the proof.

3. (i) We will prove by induction on n. Let s_n be the size of a smallest circuit computing $f : \{0,1\}^n \to \{0,1\}$. We want to show that $s_n \leq 2^{n+2} - 4$. For $n = 0$ we see that f is a constant function, thus can be computed by a circuit with $0 = 2^{0+2} - 4$ gates. For the inductive step $(n-1) \Rightarrow n$ we use the expression

$$f(x_1, \ldots, x_n) = (x_n \wedge f(x_1, \ldots, x_{n-1}, 1)) \vee (\neg x_n \wedge f(x_1, \ldots, x_{n-1}, 0)).$$

By the induction hypothesis the functions

$$f_i : \{0,1\}^{n-1} \to \{0,1\}, \quad (x_1, \ldots, x_{n-1}) \mapsto f(x_1, \ldots, x_{n-1}, i), \quad i \in \{0,1\}$$

can be computed by circuits of size at most s_{n-1}. Therefore we find a circuit for the above expression which has size at most

$$2s_{n-1} + 3 \leq 2(2^{n+1} - 4) + 4 = 2^{n+2} - 4.$$

(ii) Let $i \in \{0,1\}$ be such that $|f^{-1}(i)| \geq |f^{-1}(\neg i)|$. A constant circuit C (having zero gates) which computes $C(x) = i$ for all inputs $x \in \{0,1\}^n$ does the job.

4. Assume, towards contradiction, that $\mathbb{P}_{x \sim P_1}[T(x,y) = 1] \leq \frac{1}{2} + \delta$ for all $y \in \{0,1\}^{n_2}$. Then we have

$$\mathbb{P}_{x \sim P_1, y \sim P_2}[T(x,y) = 1] = \sum_{(x,y)\,:\,T(x,y)=1} P_1(x) P_2(y)$$

$$= \sum_y P_2(y) \sum_{x\,:\,T(x,y)=1} P_1(x) = \sum_y P_2(y) \cdot \mathbb{P}_{x \sim P_1}[T(x,y) = 1]$$

$$\leq \left(\frac{1}{2} + \delta\right) \cdot \sum_y P_2(y) = \frac{1}{2} + \delta,$$

a contradiction, finishing the proof.

Siehe nächstes Blatt!

5. (i) The inequalities

$$P(\omega) \leq 2^{-k} \quad \text{for all} \quad \omega \in \Omega$$

are equivalent to the inequalities

$$-\log P(\omega) \geq k \quad \text{for all} \quad \omega \in \Omega \text{ such that } P(\omega) > 0.$$

(ii) Assume, without loss of generality, that $H_\infty(P_1) \geq H_\infty(P_2)$. For any $y \in \{0,1\}^n$, we have

$$\begin{aligned}
\mathbb{P}_{x \sim P_1 + P_2}[x = y] &= \sum_{x_1 \in \{0,1\}^n} \mathbb{P}_{z_1 \sim P_1}[z_1 = x_1] \cdot \mathbb{P}_{z_2 \sim P_2}[z_2 = y - x_1] \\
&\leq 2^{-H_\infty(P_1)} \cdot \sum_{x_1 \in \{0,1\}^n} \mathbb{P}_{z_2 \sim P_2}[z_2 = y - x_1] \\
&= 2^{-H_\infty(P_1)}.
\end{aligned}$$

Therefore $H_\infty(P_1 + P_2) \geq H_\infty(P_1)$, concluding the proof.