Topics in Discrete Mathematics

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Assignment 1

To be completed by October 16

Solution of every problem should be no longer than one page!

Problem 1: A student is taking a combinatorics course. While studying, she solves at least 1 problem every day, and solves a total of 46 problems in October. Show that for every $1 \leq k \leq 15$, there is a period of consecutive days over which she solves exactly k problems. Must such a period exist for $k = 16$?

Problem 2: Prove that for any k-colouring of $\binom{\mathbb{N}}{2}$ there exists an infinite $S \subset \mathbb{N}$ such that $\binom{S}{2}$ is monochromatic. Deduce the finite version of Ramsey's Theorem from this.

Problem 3: At a university, student ID numbers have five digits, each between 0 and 9. To prevent clerical errors, each pair of ID numbers must have at least two differences between them. What is the maximum number of students the university can have?

Problem 4: If G is a graph, a *walk* on the graph is a sequence of vertices $x_1, x_2, \ldots, x_m$, not necessarily distinct, such that $\{x_i, x_{i+1}\} \in E(G)$ for all $1 \leq i \leq m-1$, and no edge is repeated. Suppose G is a connected graph on n vertices with k vertices of odd degree. Show that the edges of G can be covered by $\max\left\{1, \frac{k}{2}\right\}$ edge-disjoint walks. Can they be covered by fewer?

Problem 5: Show that if $S \subset \{1, 2, \ldots, 2n\}$ has n elements, without one being a proper factor of another, then $\min S \geq 2^{\lfloor \log_3(2n) \rfloor}$.

Problem 6: Show that if the complete graph K_n is partitioned into smaller cliques in such a way that every edge is covered exactly once, then at least n cliques are needed. Deduce that n non-collinear points in $\mathbb{R}^2$ determine at least n lines.

Problem 7: A graph G is said to be (n, d, λ, μ)-*strongly regular* if it has n vertices, each of degree d, any two adjacent vertices have exactly λ common neighbours, and two non-adjacent vertices have exactly μ common neighbours. Prove that we must have $(n-d-1)\mu = d(d-\lambda-1)$.