Topics in Discrete Mathematics

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Assignment 2

To be completed by October 30

Solution of every problem should be no longer than one page!

Problem 1: Given an arbitrary subset $S \subset \mathbb{Z}^2$, and two finite subsets $A, B \subset \mathbb{Z}$, define the density of S relative to the combinatorial rectangle $A \times B$ to be $d_S(A,B) = \frac{|S \cap (A \times B)|}{|A||B|}$. Let $d(S) = \lim_{k \to \infty} \sup\{d_S(A,B) : A, B \subset \mathbb{Z}, |A| = |B| = k\}$. Show that $d(S) \in \{0,1\}$.

Problem 2: Let F be a graph on f vertices with chromatic number $\chi(F) = r$. Show that for every $\varepsilon > 0$, there is a $\delta > 0$ and an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, an n-vertex graph G with $\left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2}$ edges must contain at least $\delta\binom{n}{f}$ copies of F.

Problem 3: Show that for $k \geq 2$, if G is an n-vertex graph with at least $n^{1+1/k}$ edges, then G contains a cycle of length at most $2k$.

Problem 4: A *k-matching* is a collection of k pairwise-disjoint edges. Show that a bipartite graph on vertex classes U and V, $|U| = |V| = n$, with no $(k+1)$-matching has at most kn edges. Can you classify all extremal graphs?

Problem 5: Let $x_1, x_2, \ldots, x_{3p} \in \mathbb{R}^2$ be such that $|x_i - x_j| \leq 1$ for all i, j. Show that there are at most $3p^2$ pairs $\{i, j\}$ such that $|x_i - x_j| > \frac{\sqrt{2}}{2}$. Is this bound tight?

Problem 6:

(a) Let G be a graph on n vertices with at most one loop (an edge from a vertex to itself) per vertex. Show that if G has no trail (a walk without repeated edges) of length 3, then G has at most n edges.

(b) Suppose we have integers $1 < a_1 < a_2 < \ldots < a_k \leq N$ such that no a_i divides the product of any two others. Show that $k \leq \pi(N) + N^{2/3}$, where $\pi(N)$ is the number of primes less than or equal to N.