Topics in Discrete Mathematics

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Assignment 3

To be completed by November 13

Solution of every problem should be no longer than one page!

Problem 1: Let $\mathcal{H}$ be a 3-uniform hypergraph with n vertices and m edges. A set of vertices $S \subset V(\mathcal{H})$ is said to be *independent* if no edge of $\mathcal{H}$ is contained in S. Show that $\mathcal{H}$ contains an independent set of size $\frac{2n\sqrt{n}}{3\sqrt{3m}}$.

Problem 2: Recall that the Ramsey number $R(s,t)$ is the minimal n such that any n-vertex graph contains either a clique of size s or an independent set of size t. Show that if there are $n \in \mathbb{N}$ and $p \in [0,1]$ such that $\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, then $R(s,t) > n$. Deduce that $R(4,t) = \Omega\left(\left(\frac{k}{\ln k}\right)^{3/2}\right)$.

Problem 3: A *binary code* is a collection of $\{0,1\}$-strings, and is said to be *prefix-free* if no one string appears as a prefix in another. Show that if N_i is the number of strings of length i in a prefix-free code, then $\sum_i N_i 2^{-i} \leq 1$.

Problem 4: A *cut* of a graph G is a bipartition of the vertices $V(G) = U \cup W$, and the edges of the cut $E(U,W)$ are those with one endpoint in each part. Show that any n-vertex graph G with m edges has a cut with at least $m/2$ edges.

Problem 5:

(a) Show that any tournament on $2^{k+1} - 2$ vertices contains a set of k vertices that is not dominated by any other vertex.

(b) Show that in fact the same is true for any tournament on $k2^{k-1}$ vertices. [Note: This may be challenging!]