Topics in Discrete Mathematics

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Assignment 4

To be completed by November 27

Solution of every problem should be no longer than one page!

Problem 1: The LYM inequality states that for any antichain $\mathcal{F}$ of subsets of $[n]$, we have $\sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$. Show that we have equality if and only if $\mathcal{F}$ consists of all subsets of size k for some $0 \leq k \leq n$.

Problem 2: Let $\mathcal{F}$ be a family of subsets of $[n]$ without three distinct sets $F_1 \subset F_2 \subset F_3$. Show that $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lceil \frac{n+1}{2} \rceil}$.

Problem 3: Given a set X of size n, we say that a family of subsets $\mathcal{F}$ *separates* X if for every $x \in X$ and $y \in X$, $y \neq x$, there is some set $F \in \mathcal{F}$ with $x \in F$ and $y \notin F$. Show that if $\mathcal{F}$ separates X and $m = |\mathcal{F}|$, then $\binom{m}{\lfloor \frac{m}{2} \rfloor} \geq n$. What is the smallest separating family you can find?

Problem 4: Suppose in a town of n people there are m clubs, with the property that the number of people in a club is never a multiple of six, but the number of people common to any two clubs is divisible by six. Show that $m \leq 2n$.

Problem 5: Suppose we have m pairs (A_i, B_i) of subsets of $[n]$ with the property that $|A_i \cap B_i|$ is odd for every i, but $|A_i \cap B_j|$ is even for every $i < j$. Show that $m \leq n$.

Problem 6: Let $\mathcal{F}$ be a collection of subsets of $[n]$ such that all members of $\mathcal{F}$ and their pairwise intersections have even size. Prove that $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$.