Topics in Discrete Mathematics

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Assignment 5

To be completed by December 18

Solution of every problem should be no longer than one page!

Problem 1: Let A be a two-distance subset of $S^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$. Show that $|A| \leq \frac{1}{2}n(n+3)$.

Problem 2: Find the number of spanning trees of $K_5 \setminus \{e\}$; that is, the complete graph on five vertices with one edge removed. Can you determine the number of spanning trees in $K_n \setminus \{e\}$ for general n?

Problem 3*: Cayley's formula gives the number of labelled trees (in other words, relabelling the vertices of a tree produces a different tree). A *rooted* tree is a labelled tree with one special vertex identified as the *root* of the tree. A *rooted forest* is a vertex-disjoint union of rooted trees. In other words, it is a cycle-free graph on vertices $[n]$ such that every connected component has a root vertex.

(a) How many rooted trees are there on n vertices?

(b) How many rooted forests are there on n vertices?

Problem 4: Suppose G is a connected graph with largest eigenvalue λ_1.

(a) Show that the coordinates of any λ_1-eigenvector must all be non-zero with the same sign.

(b) Deduce that the λ_1-eigenspace has dimension 1.

Problem 5: Show that a connected graph with maximum eigenvalue λ_1 is bipartite if and only if $-\lambda_1$ is also an eigenvalue.

Problem 6*: Suppose G is a graph with k distinct eigenvalues. Show that the diameter of G is at most $k-1$.