Topics in Discrete Mathematics

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Practice Problems

Solution of every problem should be no longer than one page!

Problem 1: Show that if G is a graph with even degrees, the edges of G can be oriented in such a way that every vertex in the resulting orientation has the same in-degree as out-degree.

Problem 2: Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Show there is a set of consecutive numbers $a_k, a_{k+1}, \ldots, a_\ell$ whose sum $\sum_{i=k}^{\ell} a_i$ is divisible by n.

Problem 3: Prove that for every $k \geq 2$ there exists an $n_0 = n_0(k)$ such that every colouring of $1, 2, \ldots, n_0$ in k colours contains three distinct numbers $1 \leq a, b, c \leq n_0$ satisfying $a \cdot b = c$ that have the same colour.

Problem 4: Prove that for every positive integer r there exists $N(r)$ such that for all $n \geq N(r)$, any colouring of all subsets of $[n]$ into r colours contains two non-empty disjoint sets X and Y such that X, Y and $X \cup Y$ have the same colour.

Problem 5: A *transitive tournament* is an orientation of a complete graph for which the vertices can be numbered in such a way that (i, j) is a directed edge if and only if $i < j$.

(i) Show that every orientation of the complete graph K_n contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices.

(ii) Show that if $k \geq 2 \log_2 n + 2$ there is an orientation of K_n with no transitive tournament on k vertices.

Problem 6: Let $g_1(x), \ldots, g_k(x)$ be bounded real functions and let $f(x)$ be another real function. Suppose there are positive constants ε and δ such that if $f(x) - f(y) > \varepsilon$, then $\max_i (g_i(x) - g_i(y)) > \delta$. Prove that f is also bounded.

Problem 7: Prove that every set of $2^m + 1$ vectors in $\mathbb{R}^m$ with integer coordinates contains a pair of vectors whose average also has integer coordinates.

Problem 8: Let G be a graph with n vertices and m edges. Show that G has at least

$$\frac{4m}{3n}\left(m-\frac{n^2}{4}\right)$$

triangles. Moreover, show this estimate is tight (best possible) when $m = n^2/3$.

Problem 9: Let G be a graph on n vertices and let $\overline{G}$ be its complement. Let $t(G)$ denote the total number of triangles in G and $\overline{G}$. Express $t(G)$ as a function of the degrees $d_1, \ldots, d_n$ of the vertices of G and prove that

$$t(G) \geq \frac{n(n-1)(n-5)}{24}.$$

Problem 10: Suppose $r \geq 3$, $n \geq r+1$, and let $\mathrm{ex}(n, K_r)$ denote the maximum number of edges in a graph on n vertices that does not contain K_r as a subgraph. Show that any graph G on n vertices with at least

$$\mathrm{ex}(n, K_r) + 1$$

edges must contain $K_{r+1} - e$; i.e., a copy of a clique of size $r+1$ with one missing edge.

Problem 11: Let H be an r-uniform (edge edge has size r) hypergraph on n vertices with $m = cn^{r-1/t^{r-1}}$ edges. Show that for a sufficiently large constant c, H contains a collection of r disjoint sets $U_1, \ldots, U_r$ such that $|U_i| = t$ for all i and all r-tuples intersecting each U_i in one vertex are edges of H. (That is, H contains a complete r-partite subhypergraph with parts of size t.)

Problem 12: Let X be a set of n points in the plane. Prove that the number of pairs $x_i, x_j \in X$ such that the distance between x_i and x_j equals 1 is at most $cn^{3/2}$ for some absolute constant c.

Problem 13: Let D be a directed graph on n vertices such that the outdegree of every vertex is larger than $\log_2 n$. Prove that D contains an even directed cycle.

[Hint: Show that the vertices of D can be partitioned into two parts such that every vertex from one part has an out-neighbour in the other.]

Problem 14: Let $\mathcal{F}$ be a collection of subsets of X such that every two members of $\mathcal{F}$ intersect in at least two points. Prove that the vertices of X can be 2-coloured so that no set in $\mathcal{F}$ is monochromatic.

Problem 15: Let $k \geq 4$ and let $\mathcal{F}$ be a collection of k-element subsets of X. Prove that if $\mathcal{F}$ has fewer than $\frac{4^{k-1}}{3^k}$ sets then the vertices of X can be 4-coloured so that every set in $\mathcal{F}$ is *rainbow*; i.e., contains vertices of all 4 colours.

Problem 16: Let G be a graph with average degree at least $2d$. Prove that G contains a non-empty subgraph G' with minimum degree at least d. Using this, show that if G has n vertices and kn edges, then it contains every tree on k vertices as a subgraph (a tree is a connected graph with no cycles).

Problem 17: Given a tournament T, a *Hamiltonian path* is a directed path that visits every vertex of T exactly once.

(i) Prove that every tournament contains a Hamiltonian path.

(ii) Prove there exists a tournament T on n vertices which contains at least $n!2^{-(n-1)}$ distinct Hamiltonian paths.

Problem 18: Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$ of unit length $|v_i| = 1$. Prove there are signs $\varepsilon_i = \pm 1$ such that

$$|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| \leq \sqrt{n}.$$

Show that this is tight; i.e., the $\sqrt{n}$ estimate cannot be improved.

Problem 19: Given a hypergraph H, the transversal number $\tau(H)$ is the minimal cardinality of a set of vertices which intersects all edges of H. Prove that if H has n vertices and m edges all of size r, then for any $p \in [0,1]$,

$$\tau(H) \leq pn + (1-p)^r m.$$

Deduce from this that

$$\tau(H) \leq \frac{m + n \log r}{r},$$

where log is the natural log to the base e.

Problem 20: Let $\{(A_i, B_i) : 1 \leq i \leq h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ and $|B_i| = \ell$ for all i, $A_i \cap B_i = \emptyset$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that $h \leq \frac{(k+\ell)^{k+\ell}}{k^k \ell^\ell}$.

Problem 21: A *sunflower* with k petals and core Y is a collection of sets $S_1, \dots, S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$ and the sets $S_i - Y$ are all non-empty. Let $\mathcal{F}$ be a family of sets each of cardinality s. Prove that if $|\mathcal{F}| > s!(k-1)^s$ then $\mathcal{F}$ contains a sunflower with k petals.

Problem 22: Let G_1 and G_2 be two graphs on the same vertex set V. Prove that the chromatic number of the union $G_1 \cup G_2$ (we take the union of the edge sets of both graphs) satisfies

$$\chi(G_1 \cup G_2) \leq \chi(G_1) \cdot \chi(G_2).$$

Use this to show that if $H_1, \dots, H_t$ are bipartite graphs whose union is a complete graph on n vertices, then $t \geq \log_2 n$.

Problem 23: Let $A_1, \dots, A_m$ be subsets of an n-element set. Assume that their pairwise symmetric differences $A_i \Delta A_j = (A_i - A_j) \cup (A_j - A_i)$ have only two sizes. Prove that $m \leq \frac{n(n+1)}{2} + 1$. Find $m = \frac{n(n-1)}{2} + 1$ subsets of an n-element set with only two sizes of symmetric differences.

Problem 24: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an n-element set with the property that $|A \cap B|$ is *odd* for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

Problem 25: Let λ_1 be the maximum eigenvalue of a graph G. Prove that the chromatic number of G is at most $\lambda_1 + 1$.

Problem 26: Suppose that a connected graph G has only two distinct eigenvalues. Prove that G is the complete graph.

[Hint (to be read backwards): ?hparg eht fo retemaid eht tuoba yas uoy nac tahW]

Problem 27: Let G be a graph on n vertices with m edges. Let $L(G)$ be the line graph of G.

(i) Show that all eigenvalues of $L(G)$ satisfy $\lambda_i \geq -2$.

(ii) Show that if $m > n$, then the smallest eigenvalue satisfies $\lambda_m = -2$.