Prüfung Lin Alg. & Num. Math. Winter 2012

1. a)

$$A \cdot B(:,2) = \begin{pmatrix} b+3t \\ 2 \\ bt+1 \end{pmatrix}$$

$$B(3,:) \cdot A = (1, \; bt, \; 2+b)$$

b)

$$\left(\begin{array}{ccc|c} 1 & 1 & 3t & 1 \\ 1 & 0 & 2 & 1 \\ 0 & t & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 3t & 1 \\ 0 & -1 & 2-3t & 0 \\ 0 & t & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 3t & 1 \\ 0 & -1 & 2-3t & 0 \\ 0 & 0 & 1+2t-3t^2 & 1 \end{array}\right)$$

A invertierbar $\Leftrightarrow \quad 1+2t-3t^2 \neq 0 \quad \Leftrightarrow \quad t \in \mathbb{R} \setminus \{1, -1/3\}$

$$t_{1,2} = \frac{-2 \pm \sqrt{4-4\cdot(-3)}}{2\cdot(-3)} = \frac{-2 \pm 4}{-6} = 1 \,/\, -1/3$$

$$x_3 = \frac{1}{1+2t-3t^2}, \quad x_2 = (2-3t)\,x_2 = \frac{2-3t}{1+2t-3t^2},$$

$$x_1 = 1 - x_2 - 3t\,x_3 = \frac{-1+2t-3t^2}{1+2t-3t^2}$$

$$\Rightarrow x = \frac{1}{1+2t-3t^2} \begin{pmatrix} -1+2t-3t^2 \\ 2-3t \\ 1 \end{pmatrix} = A^{-1} c$$

2.

(10)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ -2 \\ 3 \\ 2 \\ 1 \\ -2 \end{pmatrix},$$

$$A^T A = \begin{pmatrix} 8 & 0 & 2 \\ 0 & 8 & 4 \\ 2 & 4 & 8 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 2 \\ 12 \\ 14 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 8 & 0 & 0 & 6 \\ 0 & 8 & 6 & 14 \\ 0 & 6 & 8 & 8 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 4 & 0 & 0 & 3 \\ 0 & 4 & 3 & 7 \\ 0 & 3 & 4 & 4 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 4 & 0 & 0 & 3 \\ 0 & 4 & 3 & 7 \\ 0 & 0 & 4-\frac{9}{4} & 4-\frac{7\cdot 3}{4} \end{array}\right)$$

$$4-\frac{9}{4} = \frac{16-9}{4} = \frac{7}{4} \qquad 4-\frac{7\cdot 3}{4} = \frac{16-21}{4} = -\frac{5}{4}$$

$\Rightarrow a = 3/4$

$\Rightarrow c = -\frac{5}{7}$

$\Rightarrow b = \frac{1}{4}\left(7 - 3\cdot\frac{-5}{7}\right) = \frac{49+15}{4\cdot 7} = \frac{64}{4\cdot 7} = \frac{16}{7}$

$a = \frac{3}{4}, \quad b = \frac{16}{7}, \quad c = -\frac{5}{7}$

$$\left(\begin{array}{ccc|c} 4 & 0 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 2 & 4 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 7 \\ 0 & 2 & 1 & 3 \\ 0 & +8 & +15 & +27 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 7 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 11 & 15 \end{array}\right)$$

$\Rightarrow c = \frac{15}{11}$

$\Rightarrow b = \frac{1}{2}\left(3 - \frac{15}{11}\right) = \frac{1}{2}\,\frac{33-15}{11} = \frac{9}{11}$

$\Rightarrow a = 7 - 4\cdot\frac{15}{11} - 2\cdot\frac{9}{11} = \frac{1}{11}(77-60-18) = \frac{-1}{11}$

$\underline{\underline{a = \frac{-1}{11}, \quad b = \frac{9}{11}, \quad c = \frac{15}{11}}}$

Taschenrechner: keine Punkte

3. a) $\lambda_1 = -1, \quad \lambda_2 = -3 \quad \Rightarrow \quad \det(A-\lambda I) = c(\lambda+1)(\lambda+3)(\lambda-\lambda_3)$

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 2 & 6 \\ -2 & -3-\lambda & -2 \\ -3 & -1 & -6-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-3-\lambda)(-6-\lambda) + 2(-2)(-3) + 6(-2)(-1)$$

$$- (3-\lambda)(-2)(-1) - 2(-2)(-6-\lambda) - 6(-3)(-3-\lambda)$$

$$= (\lambda+3)\cdot[-(3-\lambda)(-6-\lambda) - 18] + 12 + 12 + 2(\lambda-3) - 4(\lambda+6)$$

$$= (\lambda+3)(\cancel{18} - 3\lambda - \lambda^2 - \cancel{18}) + \cancel{24} + \underbrace{2\lambda - 6 - 4\lambda}_{= -2\lambda - 6 = -2(\lambda+3)} - \cancel{24}$$

$$= (\lambda+3)(-\lambda^2 - 3\lambda - 2)$$

$$= -(\lambda+3)(\lambda+1)\cdot(\lambda+2) \quad \Rightarrow \quad \underline{\underline{\lambda_3 = -2}}$$

$$\| $$

$-\lambda^3 - 6\lambda^2 - 11\lambda - 6 \rightarrow$ Von dieser Formel direkt zur faktorisie-
ten Form $\Rightarrow$ TR $\Rightarrow$ nur

b) $\lambda_1 = -1$:

3

$$\left(\begin{array}{ccc|c} 4 & 2 & 6 & 0 \\ -2 & -2 & -2 & 0 \\ -3 & -1 & -5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ \cancel{0} & \cancel{-2} & \cancel{2} & \cancel{0} \end{array}\right)$$

$$\Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

$\lambda_2 = -3$:

$$\left(\begin{array}{ccc|c} 6 & 2 & 6 & 0 \\ -2 & 0 & -2 & 0 \\ -3 & -1 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 3 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$\lambda_3 = -2$:

$$\begin{array}{ccc|c} 5 & 2 & 6 & 0 \\ -2 & -1 & -2 & 0 \\ -3 & -1 & -4 & 0 \end{array} \rightsquigarrow \begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ -2 & -1 & -2 & 0 \\ 3 & 1 & 4 & 0 \end{array}$$

$$\rightsquigarrow \begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \qquad \Rightarrow v_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

c)

$$U = \begin{pmatrix} -2 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$AU = UD \Rightarrow A = UDU^{-1}$$

$$\ddot{x}(t) = Ax(t) = UDU^{-1}x(t) \Leftrightarrow U^{-1}\ddot{x}(t) = DU^{-1}x(t)$$

$$x(t) = U \cdot y(t), \qquad \ddot{y}(t) = D\,y(t)$$

$$y_1(t) = a_1 \cos t + b_1 \sin t$$

$$y_2(t) = a_2 \cos\sqrt{3}t + b_2 \sin\sqrt{3}t$$

$$y_3(t) = a_3 \cos\sqrt{2}t + b_3 \sin\sqrt{2}t$$

$$\Rightarrow x(t) = U \cdot y(t) = v_1\, y_1(t) + v_2\, y_2(t) + v_3\, y_3(t)$$

$$= (a_1 \cos t + b_1 \sin t)\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + (a_2 \cos\sqrt{3}t + b_2 \sin\sqrt{3}t)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (a_3 \cos\sqrt{2}t + b_3 \sin\sqrt{2}t)\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

4.

a) $\vec{x}_{k+1} = \vec{x}_k - J(\vec{x}_k)^{-1} F(\vec{x}_k)$, $\quad \vec{x}_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^3 - 4xy + y^3 - 1 \\ x^2y^2 - x - y - 2 \end{pmatrix}$$

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 - 4y & 3y^2 - 4x \\ 2xy^2 - 1 & 2yx^2 - 1 \end{pmatrix}$$

b) $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow F(\vec{x}_0) = \begin{pmatrix} 8 - 8 + 1 - 1 \\ 4 - 2 - 1 - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$,

$$J(\vec{x}_0) = \begin{pmatrix} 8 & -5 \\ 3 & 7 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 8 & -5 & 0 \\ 3 & 7 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 24 & -15 & 0 \\ 24 & 56 & -8 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 24 & -15 & 0 \\ 0 & 71 & -8 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 8 & -5 \\ 3 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{71} \\ -\frac{8}{71} \end{pmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{pmatrix} 2 + 5/71 \\ 1 + 8/71 \end{pmatrix} \approx \begin{pmatrix} 2.0704 \\ 1.1127 \end{pmatrix}$$

c) $x_0 = y_0 = a$

$$\Rightarrow F(\vec{x}_0) = \begin{pmatrix} 2a^3 - 4a^2 - 1 \\ a^4 - 2a - 2 \end{pmatrix}$$

$$\Rightarrow J(\vec{x}_0) = \begin{pmatrix} 3a^2 - 4a & 3a^2 - 4a \\ 2a^3 - 1 & 2a^3 - 1 \end{pmatrix}$$

<u>nicht invertierbar</u> $\Rightarrow$ Newton-Schritt nicht durchführbar!

6. a)

$$g(-h) = \frac{f(x_0-h) - f(x_0+h)}{-2h} = \frac{f(x_0+h) - f(x_0-h)}{2h} = g(h)$$

b)

$$g(h) = a_0 + a_1 h^2 + a_2 h^4 + a_3 h^6 + O(h^8)$$

$f(x) = 2^x$, $x_0 = 0$

$$g(1) = \frac{2 - 2^{-1}}{2} = 1 - 1/4 = \frac{3}{4} \,\hat{=}\, 0.75$$

$$g(\tfrac{1}{2}) = \frac{2^{1/2} - 2^{-1/2}}{2 \cdot 1/2} = \sqrt{2} - 1/\sqrt{2} = \frac{2-1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \approx 0.7071$$

$$g(\tfrac{1}{4}) = \frac{2^{1/4} - 2^{-1/4}}{2 \cdot 1/4} = 2\left(\sqrt[4]{2} - \frac{1}{\sqrt[4]{2}}\right) = 2 \cdot \frac{\sqrt{2}-1}{\sqrt[4]{2}} \approx 0.6966$$

$$= -2^{1/4}(\sqrt{2}-2)$$

c) Interpolation: Aitken-Neville

1	$0.75 = p_1$	$\frac{(0-1)p_2 - (0-1/4)p_1}{(1/4-1)} = p_{12} \approx 0.6928$	
1/4	$0.7071 = p_2$		p_{123}
1/16	$0.6966 = p_3$	$\frac{(0-1/4)p_3 - (0-1/16)p_2}{(1/16-1/4)} = p_{23} \approx 0.6931$	

$$p_{123} = \frac{(0-1)p_{23} - (0-1/16)p_{12}}{(1/16-1)} \approx 0.6931 \approx g(0) = f'(0)$$

alternativ: Lagrange-Interpolation

d) rel. Fehler: $\frac{|\ln 2 - p_{123}|}{|\ln 2|} = 3.46 \cdot 10^{-7}$

c) alternativ: Lagrange-Interpolation

$$l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \quad p(x) = f(x_1)\, l_1(x) + f(x_2)\, l_2(x) + f(x_3)\, l_3(x)$$

$x_1 = 1$, $x_2 = 1/4$, $x_3 = 1/16$

$$l_1(x) = \frac{64}{45}\left(x^2 - \frac{5}{16}x + \frac{1}{64}\right), \quad l_1(0) = 1/45$$

$$l_2(x) = -\frac{64}{9}\left(x^2 - \frac{17}{16}x + \frac{1}{16}\right), \quad l_2(0) = -4/9$$

$$l_3(x) = \frac{256}{45}\left(x^2 - \frac{5}{4}x + \frac{1}{4}\right), \quad l_3(0) = \frac{64}{45}$$

$$p(0) = \frac{1}{45} \cdot \frac{3}{4} + \left(-\frac{4}{9}\right)\frac{1}{\sqrt{2}} + \frac{64}{25}(2-\sqrt{2})\,2^{1/4} \approx 0.6931$$

5. $$I = \int_0^1 x^{3/2} f(x)\,dx, \quad \hat{I} = w_1 f(q) + w_2 f(1), \quad 0<q<1$$

a) $f(x)=1: \int_0^1 x^{3/2}\cdot 1\,dx = [x^{5/2}\cdot\frac{2}{5}]_0^1 = \frac{2}{5} \overset{!}{=} w_1\cdot 1 + w_2\cdot 1$

$f(x)=x: \int_0^1 x^{3/2}\cdot x\,dx = [x^{7/2}\cdot\frac{2}{7}]_0^1 = \frac{2}{7} \overset{!}{=} w_1 q + w_2\cdot 1$

$f(x)=x^2: \int_0^1 x^{3/2}\cdot x^2\,dx = [x^{9/2}\cdot\frac{2}{9}]_0^1 = \frac{2}{9} \overset{!}{=} w_1 q^2 + w_2\cdot 1^2$

b) $\begin{cases} w_1+w_2 = 2/5 & (I) \\ w_1 q + w_2 = 2/7 & (II) \end{cases} \Rightarrow$ (I) - (II): $w_1(1-q) = 2/5 - 2/7 = \frac{2(7-5)}{35} = \frac{4}{35}$

$\Rightarrow w_1 = \frac{4}{35}\cdot\frac{1}{1-q}$

$\Rightarrow w_2 = \frac{2}{5} - \frac{4}{35}\frac{1}{1-q}$

c) $w_1 q^2 + w_2 = \frac{2}{9}$

$\Rightarrow w_1 q^2 + \frac{2}{5} - w_1 = \frac{2}{9} \Rightarrow w_1(1-q^2) = w_1(1-q)(1+q) = -\frac{2}{9}+\frac{2}{5} = -\frac{2(5-9)}{45} = \frac{+8}{45}$

$\Rightarrow \frac{4}{35}\cdot\frac{1}{\cancel{1-q}}\cancel{(1-q)}(1+q) = \frac{8}{45}$

$\Rightarrow q+1 = \frac{2\cancel{8}\cdot\cancel{35}\,7}{\cancel{4}\cdot\cancel{45}\,9} = \frac{14}{9} \Rightarrow \underline{\underline{q = \frac{5}{9}}}$

$\Rightarrow \underline{\underline{w_1}} = \frac{4}{35}\cdot\frac{1}{1-5/9} = \frac{4}{35}\cdot\frac{9}{4} = \underline{\underline{\frac{9}{35}}}$

$\Rightarrow \underline{\underline{w_2}} = \frac{2}{5} - w_1 = \frac{2\cdot 7}{35} - \frac{9}{35} = \frac{5}{35} = \underline{\underline{\frac{1}{7}}}$

d) $\hat{I} = w_1 f(q) + w_2 f(1), \qquad f(x) = \frac{1}{1+x}$

$= \frac{9}{35}\cdot\frac{1}{1+5/9} + \frac{1}{7}\cdot\frac{1}{1+1}$

$= \frac{9}{35}\cdot\frac{9}{14} + \frac{1}{14} = \frac{1}{14}\left(\frac{116}{35}\right) = \frac{58}{245} \approx 0.2367$

Rel. Fehler $\frac{|I-\hat{I}|}{|I|} = 3.067\cdot 10^{-3}$