Problem Sheet 6

1. We have three metal alloys M_1, M_2 and M_3, all of which are composed of copper, silver and gold in different proportions. These percentages are given in the following table

	Copper	Silver	Gold
M_1	20	60	20
M_2	70	10	20
M_3	50	50	0

Determine in which ratios should the alloys M_1, M_2 and M_3 be mixed in order to create e a fourth alloy which is formed out of 40% copper, 50% silver and 10% gold.

Hint: To solve this problem you will need to create and solve a linear system of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$, for some $\mathbf{A}$ and $\mathbf{b}$.

2. Solve the following problems using Gaussian elimination.

a) Let $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 5 & -1 \\ -3 & -8 & 4 \end{pmatrix}$.

1. Find the rank of $\mathbf{A}$.
2. Let $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ -10 \end{pmatrix}$. Find the solution of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$, if the solution exists.

b) Consider a matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & -1 \\ k & 5 & -4 \end{pmatrix}$, where $k \in \mathbb{R}$ is a parameter.

1. Find the rank of $\mathbf{A}$, depending on the parameter k.
2. Let $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$. Find the solution, if it exists, of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$, depending on k.

c) Consider a matrix $\mathbf{A} = \begin{pmatrix} 1 & m & 1 \\ 1 & 1 & 1 \\ m & 1 & m-1 \end{pmatrix}$, where $m \in \mathbb{R}$ is a parameter.

1. Find the rank of $\mathbf{A}$, depending on m.
2. Let $\mathbf{b} = \begin{pmatrix} 1 \\ m+1 \\ m \end{pmatrix}$. Find the solution, if it exists, of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$, depending on m.

Bitte wenden!

3. Consider a tridiagonal, $n \times n$ matrix $\mathbf{A}$, with entries as follows

$$\begin{aligned} a_{i,i+1} &= -1 \qquad \text{for } i = 1, \ldots, n-1, \\ a_{ii} &= 2 \qquad \text{for } i = 1, \ldots, n, \\ a_{i,i-1} &= -1 \qquad \text{for } i = 2, \ldots, n \end{aligned}$$

a) Write down $\mathbf{A}$ for $n = 4$, take $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and find the solution $\mathbf{x}$ of $\mathbf{Ax} = \mathbf{b}$.

b) Create a MATLAB function

```
function x = SolverTridiag(b)
```

which takes as argument a column vector $\mathbf{b}$, creates a tridiagonal matrix $\mathbf{A}$ as described in part **a)**, and returns the solution $\mathbf{x}$ of the system $\mathbf{Ax} = \mathbf{b}$. To solve the system, you can use the backslash operator.

c) Let $n \in \mathbb{N}$ be arbitrary, and take a vector $\mathbf{b} \in \mathbb{R}^n$ to be $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ n+1 \end{pmatrix}$. Find the solution $\mathbf{x}$ of the system $\mathbf{Ax} = \mathbf{b}$, with pen and paper.

d) Create a MATLAB function

```
function x = LoopSolverTridiag(b)
```

which takes asargument a column vector $\mathbf{b}$, and computes the solution $\mathbf{x}$ of the system $\mathbf{Ax} = \mathbf{b}$ without creating the matrix $\mathbf{A}$ but by solving the system directly using loops (e.g. `for`).

4. **a)** Consider a 4×4 matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 1 & 1 & 1 & 4 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} -1 \\ -4 \\ 1 \\ -3 \end{pmatrix}$. Find the solution $\mathbf{x}$ of the system $\mathbf{Ax} = \mathbf{b}$.

b) Consider an $n \times n$ matrix $\mathbf{A} = \begin{pmatrix} d_1 & & & c_1 \\ & \ddots & & \vdots \\ & & d_{n-1} & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{pmatrix}$, that is, defined as follows

$$\begin{aligned} a_{ii} &= d_i, \quad \text{for } i = 1, \ldots, n-1, \\ a_{n,i} &= c_i, \quad \text{for } i = 1, \ldots, n, \\ a_{i,n} &= c_i, \quad \text{for } i = 1, \ldots, n \end{aligned}$$

where $d_i \neq 0$ for $i = 1, \ldots, n-1$ and all other entries in the matrix are 0. When does the linear system $\mathbf{Ax} = \mathbf{b}$ admit a unique solution?

c) Create a MATLAB function

```
function x = SpecialSolver(d, c, b)
```

which takes as arguments column vectors $\mathbf{c}$ and $\mathbf{b}$, which are of equal length, and a column vector $\mathbf{d}$, which has one entry less than $\mathbf{b}$ and $\mathbf{c}$. The return argument $\mathbf{x}$ should be the solution of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$, if such a solution exists. In order to solve the system create a sparse matrix $\mathbf{A}$, as described in part **b)** using vectors $\mathbf{c}$ and $\mathbf{d}$, and solve the system using Gaussian eliminations.

5. Find the reduced row echelon form of the following general 2×2 linear system

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where $a, b, c, d, x_1, x_2, b_1$ and b_2 are real numbers. To do so use Gaussian eliminations.

Hint: You have to distinguish some cases.

6. For each of the following subsets of the set of all polynomials of degree less or equal than 7, that is of $\mathbb{P}_7$, determine whether or not they are linearly independent in $(\mathbb{P}_7, +, \cdot)$

a) Set $\{p_1(x), p_2(x), p_3(x)\}$ where $p_1(x) = x(x-1), p_2(x) = x^2 - x^3 + 1$ and $p_3(x) = x^4 + x^2$.

b) Set $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ where $p_1(x) = (x+1)(x-1), p_2(x) = x(x+1), p_3(x) = (x^3+1)(x^3-1)$ and $p_4(x) = x - 1$

c) Set $\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$ where $p_1(x) = x(x+1), p_2(x) = x^3 + 3x + 1, p_3(x) = (x^2+1)(x^3+4), p_4(x) = -3x + 2x^3 + 2$ and $p_5(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Hint: Remember that one can define a mapping from a Basis of $\mathbb{P}_7$ to $\mathbb{R}^8$, i.e. the mapping $\{1, x, x^2, \ldots, x^7\} \to \mathbb{R}^8$, that preserves all properties of the vector space and allows to reduce calculations to $\mathbb{R}^8$. See also Section 2.1 and Section 3.2.3 from lectures.

7. Given a *invertible* matrix $\mathbf{A} \in \mathbb{R}^{n,n}$, $n \in \mathbb{N}$, and two arbitrary vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, assume that

$$1 + \mathbf{v}^T \mathbf{A}^{-1} \mathbf{u} \neq 0 \,.$$

Show that it holds

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \,.$$

Hint: Try to understand what does it mean for a matrix to be the inverse of another matrix using “Satz 3.6.B”. As soon as this is clear to you, in the remainder of the proof you only need to apply the calculation rules for the matrix product “mechanically”. Important is that you keep track of matrix products that reduce to a real number (and therefore commute with the other terms).

8. We are given two bases, $\mathcal{A}, \mathcal{B}$, for $\mathbb{R}^3$

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \right\}, \qquad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 7 \\ 6 \end{pmatrix} \right\}.$$

Bitte wenden!

a) Find the change of coordinates matrix $\mathbf{S}$ from basis $\mathcal{A}$ to $\mathcal{B}$.

b) Determine the coordinates of the vector

$$\mathbf{v} = 2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 9\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} - 8\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

with respect to the basis $\mathcal{B}$.

- **Abgabe der Serien:** Donnerstag, 07.11.2013 in der Übungsgruppe oder bis 16:00 Uhr in den Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.
- **Semesterpräsenz:** Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13. Falls keine grosse Nachfrage besteht, warten die Assistenten maximal eine halbe Stunde. Wir bitten Sie deshalb, bei Fragen so früh als möglich zu erscheinen.
- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
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