Solution 1

1. Determine whether each of the following statements are true for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ and $a, b \in \mathbb{R}$.

√ (a) $\sum_{i=1}^n (x_i + y_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i$

$(x_1 + y_1) + \ldots + (x_n + y_n) = x_1 + \ldots + x_n + y_1 + \ldots + y_n \Rightarrow$ Statement holds.

√ (b) $\sum_{i=1}^n x_i = \sum_{k=1}^n x_k = \sum_{k=1}^n x_{n+1-k}$

$$\begin{aligned} \sum_{i=1}^n x_i &= x_1 + x_2 + \ldots + x_n \\ = \sum_{k=1}^n x_k &= x_1 + x_2 + \ldots + x_n \\ = \sum_{k=1}^n x_{n+1-k} &= x_n + \ldots + x_2 + x_1 \end{aligned}$$

$\Rightarrow$ Statement holds.

(c) $\sum_{i=1}^n (ax_i + b) = a\big(\sum_{i=1}^n x_i\big) + b$

$(ax_1 + b) + \ldots + (ax_n + b) \neq a(x_1 + \ldots + x_n) + b \Rightarrow$ Statement does not hold.

(d) $\sum_{i=1}^n (x_i \cdot y_i) = \big(\sum_{i=1}^n x_i\big) \cdot \big(\sum_{i=1}^n y_i\big)$

$(x_1 y_1) + \ldots + (x_n y_n) \neq (x_1 + \ldots + x_n)(y_1 + \ldots + y_n) \Rightarrow$ Statement does not hold.

√ (e) $\sum_{i=1}^n (x_i - \frac{1}{n}\sum_{j=1}^n x_j) = 0$

$(x_1 - \frac{1}{n}c) + \ldots + (x_n - \frac{1}{n}c) = (x_1 + \ldots + x_n) - c, c = x_1 + \ldots + x_n \Rightarrow$ Statement holds.

√ (f) $\sum_{i=1}^n \sum_{j=1}^n x_i \cdot y_j = \big(\sum_{i=1}^n x_i\big) \cdot \big(\sum_{j=1}^n y_j\big)$

$$\begin{aligned} x_1y_1 + x_1y_2 + \ldots + x_1y_n + \\ x_2y_1 + x_2y_2 + \ldots + x_2y_n + \\ \ldots \\ x_ny_1 + x_ny_2 + \ldots + x_ny_n + \\ = (x_1 + \ldots + x_n)(y_1 + \ldots + y_n) \end{aligned}$$

$\Rightarrow$ Statement holds.

(g) $(a-1)\left(\sum_{i=0}^n a^i\right) = a^n - 1$

$$(a-1)\left(\sum_{i=0}^{n} a^i\right) = \sum_{i=0}^{n} a^{i+1} - \sum_{i=0}^{n} a^i \qquad = a^{n+1} + \sum_{i=1}^{n} a^i - \sum_{i=1}^{n} a^i - 1 = a^{n+1} - 1 \qquad \neq a^n - 1$$

$\Rightarrow$ Statement does not hold.

2. (a) *Solution 1.* Notice that every vector can be paired with a vector pointing in the opposite direction, for example we have $\mathbf{v}^7 = -\mathbf{v}^1$, and in general, $\mathbf{v}^{i+6} = -\mathbf{v}^i$, for $i = 1, 2, \ldots, 6$. Therefore the sum is a zero vector, that is

$$\sum_{i=1}^{12} \mathbf{v}^i = \mathbf{0}.$$

Solution 2. Vectors $\mathbf{v}^1, \ldots, \mathbf{v}^{12}$ all lie on a unit circle, with equidistant angles. Therefore, we can write the vectors as

$$\mathbf{v}^i = \left(\cos\left(\frac{\pi}{2} - \frac{i2\pi}{12}\right), \sin\left(\frac{\pi}{2} - \frac{i2\pi}{12}\right)\right)^\top = \left(\cos\left(\frac{\pi}{2} - \frac{i\pi}{6}\right), \sin\left(\frac{\pi}{2} - \frac{i\pi}{6}\right)\right)^\top,$$

for $i = 1, \ldots, 12$. Since $\cos(x-\pi) = -\cos(x)$ and $\sin(x-\pi) = -\sin(x)$, we have

$$\begin{aligned}
\mathbf{v}^{i+6} + \mathbf{v}^i &= \left(\cos\left(\frac{\pi}{2} - \frac{(i+6)\pi}{6}\right), \sin\left(\frac{\pi}{2} - \frac{(i+6)\pi}{6}\right)\right)^\top \\
&\qquad + \left(\cos\left(\frac{\pi}{2} - \frac{i\pi}{6}\right), \sin\left(\frac{\pi}{2} - \frac{i\pi}{6}\right)\right)^\top \\
&= \left(\cos\left(\frac{\pi}{2} - \frac{i\pi}{6} - \pi\right), \sin\left(\frac{\pi}{2} - \frac{i\pi}{6} - \pi\right)\right)^\top \\
&\qquad + \left(\cos\left(\frac{\pi}{2} - \frac{i\pi}{6}\right), \sin\left(\frac{\pi}{2} - \frac{i\pi}{6}\right)\right)^\top \\
&= (0,0)^\top.
\end{aligned}$$

Therefore, we reach the same conclusion as in *Solution 1.*, and we have $\sum_{i=1}^{12} \mathbf{v}^i = \mathbf{0}$.

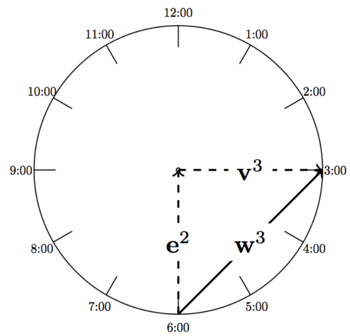
(b) We have

$$\sum_{i=1, i\neq 4}^{12} \mathbf{v}^i = \left(\sum_{i=1}^{12} \mathbf{v}^i\right) - \mathbf{v}^4 = \mathbf{0} - \mathbf{v}^4 = -\mathbf{v}^4 = \mathbf{v}^{10} = \left(\cos\left(\frac{7\pi}{6}\right), -\sin\left(\frac{7\pi}{6}\right)\right)^\top$$

with $\mathbf{v}^4$ denoting the vector pointing at $4{:}00$. Here we used the used (a) in the 2^{nd} equality.

(c) We have

$$\begin{aligned}
\sum_{i=2}^{12} \mathbf{v}^i + \frac{1}{2}\mathbf{v}^1 &= \left(\sum_{i=1}^{12} \mathbf{v}^i\right) - \mathbf{v}^1 + \frac{1}{2}\mathbf{v}^1 = \mathbf{0} - \frac{\mathbf{v}^1}{2} = -\frac{\mathbf{v}^1}{2} = \frac{\mathbf{v}^7}{2} \\
&= \left(\frac{1}{2}\cos\left(\frac{\pi}{2} - \frac{7\pi}{6}\right), \frac{1}{2}\sin\left(\frac{\pi}{2} - \frac{7\pi}{6}\right)\right)^\top \\
&= -\left(\frac{1}{2}\cos\left(\frac{\pi}{3}\right), \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right)^\top.
\end{aligned}$$



(d) We can observe that $\mathbf{w}^i = \mathbf{v}^i + (0,1)^\top$ holds for all $i = 1, \ldots, 12$. This gives us

$$\sum_{i=1}^{12} \mathbf{w}^i = \sum_{i=1}^{12} \left(\mathbf{v}^i + (0,1)^\top\right) = 12(0,1)^\top + \sum_{i=1}^{12} \mathbf{v}^i = (0,12)^\top.$$

3. (a) We do not need to check conditions 5. and 6. since they are not affected by a different addition rule.

The first rule does not hold since we have

$$(v_1, v_2)^\top \oplus (w_1, w_2)^\top = (v_1 + w_2, v_2 + w_1)^\top \quad \text{and}$$
$$(w_1, w_2)^\top \oplus (v_1, v_2)^\top = (w_1 + v_2, w_2 + v_1)^\top .$$

Thus, we would have to have $(v_1 + w_2, v_2 + w_1)^\top = (w_1 + v_2, w_2 + v_1)^\top$ for all $\mathbf{v}$ and $\mathbf{w}$, and this is not the case. Take for instance $(v_1, v_2)^\top = (1,0)^\top$ and $(w_1, w_2)^\top = (0,0)^\top$.

The second rule does not hold anymore, since

$$\mathbf{v} \oplus (\mathbf{w} \oplus \mathbf{u}) = \mathbf{v} \oplus (w_1 + u_2, w_2 + u_1) = (v_1 + w_2 + u_1, v_2 + w_1 + u_2)$$

while on the other hand

$$(\mathbf{v} \oplus \mathbf{w}) \oplus \mathbf{u} = (v_1 + w_2, v_2 + w_1) \oplus \mathbf{u} = (v_1 + w_2 + u_2, v_2 + w_1 + u_1).$$

Thus, we do not have an equality if $u_1 \neq u_2$. For example, take $\mathbf{v} = (1,0)^\top$, $\mathbf{w} = (0,1)^\top$ and $\mathbf{u} = (-1,0)^\top$. Plugging these in gives

$$\mathbf{v} \oplus (\mathbf{w} \oplus \mathbf{u}) = (1,0)^\top \text{ and } \ (\mathbf{v} \oplus \mathbf{w}) \oplus \mathbf{u} = (2,-1)^\top.$$

The third rule holds, with a zero vector being the usual zero vector $\mathbf{0} := (0,0)^\top$, because we have

$$(v_1, v_2)^\top \oplus (0,0)^\top = (v_1, v_2)^\top.$$

The fourth rule holds if for each $\mathbf{v}$ we take $-\mathbf{v} := (-v_2, -v_1)^\top$, since

$$\mathbf{v} \oplus (-\mathbf{v}) = (v_1, v_2)^\top \oplus (-v_2, -v_1)^\top = (v_1 - v_1, v_2 - v_2)^\top = (0,0)^\top.$$

The seventh rule holds since on hand we have

$$\alpha \cdot (\mathbf{v} + \mathbf{w}) = \alpha \cdot (v_1 + w_2, v_2 + w_1)^\top = (\alpha \cdot (v_1 + w_2)^\top, \alpha \cdot (v_2 + w_1)^\top) = (\alpha \cdot v_1 + \alpha \cdot w_2, \alpha \cdot v_2 + \alpha \cdot w_1)^\top,$$

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while on the other hand we have

$$\alpha \cdot \mathbf{v} + \alpha \cdot \mathbf{w} = (\alpha \cdot v_1, \alpha \cdot v_2)^\top + (\alpha \cdot w_1, \alpha \cdot w_2)^\top = (\alpha \cdot v_1 + \alpha \cdot w_2, \alpha \cdot v_2 + \alpha \cdot w_1)^\top.$$

which are clearly equal.
The eighth rule does not hold, since

$$(\alpha + \beta) \cdot \mathbf{v} = ((\alpha + \beta) \cdot v_1, (\alpha + \beta) \cdot v_2)^\top ,$$

while on the other hand

$$(\alpha \cdot \mathbf{v}) \oplus (\beta \cdot \mathbf{v}) = (\alpha \cdot v_1, \alpha \cdot v_2)^\top \oplus (\beta \cdot v_1, \beta \cdot v_2)^\top = (\alpha \cdot v_1 + \beta \cdot v_2, \alpha \cdot v_2 + \beta \cdot v_1)^\top ,$$

which need not be equal. Take for instance $\alpha = 1, \beta = -1$ and $\mathbf{v} = (1, -1)^\top$ thus $(\alpha+\beta)\cdot\mathbf{v} = (0, 0)^\top$ while $(\alpha \cdot \mathbf{v}) \oplus (\beta \cdot \mathbf{v}) = (2, 2)^\top$.

(b) We only need to consider conditions $5., 6., 7.$ and $8.$, since the first four conditions hold true since they are not affected by the new definition of scalar multiplication.
The fifth rule does not hold, since $1 \odot \mathbf{v} = (v_1, 0)^\top$ is not equal to $\mathbf{v} = (v_1, v_2)^\top$ if $v_2 \neq 0$.
The sixth rule holds, since

$$(\alpha \cdot \beta) \odot \mathbf{v} = (\alpha \cdot \beta \cdot v_1, 0)^\top$$

while

$$\alpha \odot (\beta \odot \mathbf{v}) = \alpha \odot (\beta \cdot v_1, 0)^\top = (\alpha \cdot \beta \cdot v_1, 0)^\top.$$

Therefore, we obtain equality of the expressions above.
The seventh rule also holds. The left hand side gives

$$\alpha \odot (\mathbf{v} + \mathbf{w}) = \alpha \odot (v_1 + w_1, v_2 + w_2)^\top = (\alpha \cdot v_1 + \alpha \cdot w_1, 0)^\top$$

while the right hand side gives

$$\alpha \odot \mathbf{v} + \alpha \odot \mathbf{w} = (\alpha \cdot v_1, 0)^\top + (\alpha \cdot w_1, 0)^\top = (\alpha \cdot v_1 + \alpha \cdot w_1, 0)^\top .$$

Hence, the right hand side is equal to the left hand side of the seventh rule.
The eighth rule holds, since

$$(\alpha + \beta) \odot \mathbf{v} = ((\alpha + \beta) \cdot v_1, 0)^\top = (\alpha \cdot v_1, 0)^\top + (\beta \cdot v_1, 0)^\top = \alpha \odot \mathbf{v} + \beta \odot \mathbf{v}.$$

4. (a) First, let us notice that we only need to check the rules $5. - 8.$, since those are the only rules which are affected by the change in definition of scalar multiplication. Rules $5., 6.$ and $7.$ are correct since

$$\begin{aligned}
1 \odot f(x) &= f(x), \\
(\alpha \cdot \beta) \odot f(x) &= f(\alpha \cdot \beta \cdot x) = f(\alpha \cdot (\beta \cdot x)) = \alpha \odot f(\beta \cdot x) = \alpha \odot (\beta \odot f(x)) , \\
\alpha \odot (f(x) + g(x)) &= \alpha \odot ((f + g)(x)) = (f + g)(\alpha \cdot x) = f(\alpha \cdot x) + f(\beta \cdot x) \\
&= \alpha \odot f(x) + \beta \odot g(x) .
\end{aligned}$$

Rule number eight is no longer true, since $(\alpha + \beta) \odot f(x) = f((\alpha + \beta) \cdot x)$. On the other hand, we have $\alpha \odot f(x) + \beta \odot f(x) = f(\alpha \cdot x) + f(\beta \cdot x)$. Thus,

$$f((\alpha + \beta) \cdot x) = f(\alpha \cdot x) + f(\beta \cdot x)$$

would have to hold for any arbitrary function $f \in C(\mathbb{R})$, which is clearly not the case. To provide a specific counterexample, we can take for instance the function $f(x) = 1$. The left hand side gives $f((\alpha + \beta) \cdot x) = 1$, while the right hand side gives $f(\alpha \cdot x) + f(\beta \cdot x) = 2$.

(b) Analogously to part (a), we only need to check rules $1. - 4.$ and $7. - 8.$. The first rule is broken since $f(g(x)) \neq g(f(x))$ in general. For example, if $f(x) = x^2$ and $g(x) = x + 3$ then $f(g(x)) = f(x+3) = (x+3)^2$, while $g(f(x)) = g(x^2) = x^2 + 3$, which are not equal for all x.
The second rule is correct.
The third rule is correct with the zero vector defined to be $e(x) = x$. That is, $f(e(x)) = f(x)$ and $e(f(x)) = f(x)$, so we have an equality.
The fourth rule is not correct in general, since not all continuous functions are invertible on whole $\mathbb{R}$,for example $f(x) = x^2$, though the rule is true for invertible functions by taking the inverse to be exactly $f^{-1}(x)$.
The seventh rule is not true in general since the $\alpha \cdot f(g(x))$ is not equal to $\alpha \cdot f(\alpha \cdot g(x))$, e.g. take $f(x) = g(x) = x$ and $\alpha = 2$.
The eight rule is not true since the left hand side is interpreted as $(\alpha + \beta) \cdot f(x)$, while the right hand side is interpreted as $\alpha \cdot f(\beta \cdot f(x))$, which are not equal in general (take for instance $\alpha = 1, \beta = -1$ and $f(x) = 1$).

5. (a) Yes. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $v_1 = v_2$ and $w_1 = w_2$. Then we have

$$\alpha \cdot \mathbf{v} + \beta \cdot \mathbf{w} = (\alpha \cdot v_1 + \beta \cdot w_1, \alpha \cdot v_2 + \beta \cdot w_2, \alpha \cdot v_3 + \beta \cdot w_3)^\top$$

and $\alpha \cdot v_1 + \beta \cdot w_1 = \alpha \cdot v_2 + \beta \cdot w_2$. Therefore, since $\mathbf{v}, \mathbf{w}$ were arbitrary elements of the given subset, and since α and β were arbitrary scalars, it is a subspace.

(b) No, since $\alpha \cdot (v_1, v_2, v_3)^\top = \alpha \cdot (1, v_2, v_3)^\top$ is not in the set if $\alpha = \frac{1}{2}$.

(c) No, since if $\mathbf{v}$ and $\mathbf{w} \in \mathbb{R}^3$ are such that $v_1 \cdot v_2 \cdot v_3 = 0$ and $w_1 \cdot w_2 \cdot w_3 = 0$ there is no guarantee that $\mathbf{v} + \mathbf{w}$ will have that property, that is, that $(v_1 + w_1) \cdot (v_2 + w_2) \cdot (v_3 + w_3) = 0$. Take $\mathbf{v} = (1, 0, 0)^\top$ and $\mathbf{w} = (0, 1, 1)^\top$. Then both $\mathbf{v}$ and $\mathbf{w}$ are in the set, but $\mathbf{v} + \mathbf{w} = (1, 1, 1)^\top$, hence $(v_1 + w_1) \cdot (v_2 + w_2) \cdot (v_3 + w_3) = 1$

(d) Yes. Same as in (a), take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $v_1 + v_2 + v_3 = 0$ and $w_1 + w_2 + w_3 = 0$. Then we have

$$\alpha \cdot \mathbf{v} + \beta \cdot \mathbf{w} = (\alpha \cdot v_1 + \beta \cdot w_1, \alpha \cdot v_2 + \beta \cdot w_2, \alpha \cdot v_3 + \beta \cdot w_3)$$

thus,

$$(\alpha \cdot v_1 + \beta \cdot w_1) + (\alpha \cdot v_2 + \beta \cdot w_2) + (\alpha \cdot v_3 + \beta \cdot w_3) = \alpha \cdot (v_1 + v_2 + v_3) + \beta \cdot (w_1 + w_2 + w_3) = 0$$

and it is a subspace.

(e) No, since if $\mathbf{v} = (v_1, v_2, v_3)^\top \in \mathbb{R}^3$ satifies $v_1 \leq v_2 \leq v_3$, there is no guarantee that $\alpha \cdot \mathbf{v}$ does as well. Consider $\mathbf{v} = (-3, -2, -1)^\top$ and $\alpha = -1$, so that $v_1 \leq v_2 \leq v_3$, but the same is not true for $\alpha \cdot v_1, \alpha \cdot v_2$ and $\alpha \cdot v_3$.

(f) Yes, a set of all linear combinations of two vectors satisfies the conditions for a subspace by definition.

(g) Yes. Take arbitrary $\alpha, \beta \in \mathbb{R}$ and $f, g \in C(\mathbb{R})$ such that $\int_0^1 f(x)dx = 0$, $\int_0^1 g(x)dx = 0$. Then for $h(x) = \alpha \cdot f(x) + \beta \cdot g(x)$ we have

$$\int_0^1 h(x)dx = \int_0^1 (\alpha \cdot f(x) + \beta \cdot g(x))\, dx = \alpha \cdot \int_0^1 f(x)dx + \beta \cdot \int_0^1 g(x)dx = 0\,,$$

because of linearity of integration.

(h) No, since if $p(0) = 1$ and $q(0) = 1$ then for $r(x) = p(x) + q(x) \in \mathcal{P}_2$ we have

$$r(0) = p(0) + q(0) = 2\,.$$

(i) Yes. Take $p, q \in \mathcal{P}_7$ such that $p(0) = 2 \cdot p'(0)$ and $q(0) = 2 \cdot q'(0)$. For arbitrary $\alpha, \beta \in \mathbb{R}$ we hence have

$$(\alpha \cdot p + \beta \cdot q)(0) = \alpha \cdot p(0) + \beta \cdot q(0) = 2 \cdot \alpha \cdot p'(0) + 2 \cdot \beta \cdot q'(0) = 2 \cdot (\alpha \cdot f' + \beta \cdot g')\,(0) = 2 \cdot (\alpha \cdot p + \beta \cdot q)'\,(0).$$