Solutions - Problem Sheet 5

1. Look at the separate file concerning multiple choice problems.

2. **a)** We have

$$\mathbf{AB} = \begin{pmatrix} 0 & 7 \\ 0 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{AC} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

Adding them up we have $\mathbf{AB} + \mathbf{AC} = \begin{pmatrix} 3 & 8 \\ 6 & 9 \end{pmatrix}$.

On the other hand, we have $\mathbf{B} + \mathbf{C} = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}$. Multiplying by $\mathbf{A}$ from the right, we get $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 3 & 8 \\ 6 & 9 \end{pmatrix}$. Therefore, the matrices are equal to each other, that is, we have $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$.

b) We have $\mathbf{BC} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. That means that $\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as well. Conversely, we have $\mathbf{AB} = \begin{pmatrix} 0 & 7 \\ 0 & 7 \end{pmatrix}$. Multiplying by $\mathbf{C}$ from the left $(\mathbf{AB})\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. In conclusion, we have $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ in this example.

3. (i) **a)** We have

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 4 \end{pmatrix}$$

and

$$\mathbf{A}^2 - \mathbf{B}^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}.$$

b) Recall that for the case of two scalars $a, b \in \mathbb{R}$ we have $(a+b)(a-b) = a^2 - b^2$. Why is this not the case when it comes to matrices? Let us apply the distributive property; $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ to $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$. We have

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = (\mathbf{A} + \mathbf{B})\mathbf{A} - (\mathbf{A} + \mathbf{B})\mathbf{B} = \mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} - \mathbf{B}^2.$$

Hence, the previous equation is equal to $\mathbf{A}^2 - \mathbf{B}^2$ only when $\mathbf{AB} = \mathbf{BA}$, that is, only if matrices $\mathbf{A}$ and $\mathbf{B}$ commute.

c) As we have mentioned, we will have an equality if $\mathbf{C}$ and $\mathbf{D}$ commute. Here are a couple of examples of such matrices

- $\mathbf{C} = \mathbf{D}$
- $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{D}$ arbitrary

– $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{D}$ arbitrary

– $\mathbf{C} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$

(ii) We have

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 2 \\ 3 & 0 \end{pmatrix}.$$

Therefore

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \begin{pmatrix} 10 & 4 \\ 6 & 6 \end{pmatrix}.$$

On the other hand we have

$$\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 = \begin{pmatrix} 16 & 2 \\ 3 & 0 \end{pmatrix}.$$

Hence, $(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$ is indeed not equal to $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$. In order to get the correct rule, we again use the distributive properties

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = (\mathbf{A}+\mathbf{B})\mathbf{A} + (\mathbf{A}+\mathbf{B})\mathbf{B} = \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2.$$

4. We have $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{B} \in \mathbb{R}^{3\times 2}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3\times 1}$. By straightforward computation it follows

- $\mathbf{AB} = \begin{pmatrix} 23 & 26 \\ 31 & 36 \\ -6 & -8 \end{pmatrix}$.
- $\mathbf{BA}$ is not defined.
- $\mathbf{Ax} = \begin{pmatrix} -21 \\ -5 \\ -7 \end{pmatrix}$.
- $\mathbf{A}^2 = \begin{pmatrix} -4 & -19 & 1 \\ -3 & -2 & 9 \\ -2 & -3 & -10 \end{pmatrix}$.
- $\mathbf{B}^2$ is not defined.
- $\mathbf{yx}$ is not defined.
- $\mathbf{y}^\top \mathbf{x} = 12$.
- $\mathbf{xy}^\top = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 6 & -4 \\ -3 & -9 & 6 \end{pmatrix}$.
- $\mathbf{B}^\top \mathbf{y} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$.
- $\mathbf{y}^\top \mathbf{B} = (4, 6)$.

5. **a)** We have

$$\mathbf{A}_1 = \mathbf{A} - \mathbf{I}_n = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Siehe nächstes Blatt!

Now we can compute

$$\mathbf{A}_1^2 = \mathbf{A}_1 \cdot \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_1^3 = \mathbf{A}_1 \cdot \mathbf{A}_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, we also have $\mathbf{A}_1^4 = \mathbf{A}_1\mathbf{A}_1^3 = \mathbf{A}_1\mathbf{0} = \mathbf{0}$, since $\mathbf{A}_1^3$ is the zero matrix. By induction it follows that $\mathbf{A}_1^k = \mathbf{0}$ for all $k \geq 3$.

b) Since $\mathbf{A}_1^k = \mathbf{0}$ for $k \geq 3$ the formula (1), with $k = 10$ and $n = 3$, reduces to

$$\mathbf{A}^{10} = (\mathbf{I}_3 + \mathbf{A}_1)^{10} = \mathbf{I}_3 + \binom{10}{1}\mathbf{A}_1 + \binom{10}{2}\mathbf{A}_1^2.$$

In other words, we have

$$\mathbf{A}^{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 10 \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} + 45 \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 20 & 310 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. (i)

```
%% Problem 6.(i).a)
% INPUT
% n - parameter describing the size of the matrix
% OUTPUT
% Z - nxn matrix, whose non-zero entries are shaped like the letter Z

function Z = ZShaped(n)

Z = sparse(ones(n-1, 1), (1:(n-1))', ones(n-1,1), n, n) + sparse((1:n)',...
    (n:-1:1)', (1:n)', n, n)+sparse( n*ones(n-1, 1), (2:n)', ...
    n*ones(n-1,1)', n, n);
```

```
%% Problem 6.(i).b)
% INPUT
% n - parameter describing the size of the matrix
% OUTPUT
% X - nxn matrix, whose non-zero entries are shaped like the letter X

function X = XShaped(n)

X = 2*sparse((1:n)',(1:n)',ones(n,1), n, n) + 2*sparse((1:n)', ...
    (n:-1:1)',ones(n,1));

% If n is an odd number then by the previous equation the value of the
% entry in the middle of the diagonal would be doubled, so we have to reset
% it.

if mod(n,2)
    X( (n+1)/2, (n+1)/2) = 2;
end
```

Bitte wenden!

```
%% Problem 6.(i).c)
% INPUT
% n - parameter describing the size of the matrix
% OUTPUT
% T - nxn, three band matrix

function T = ThreeBand(n)

T = sparse( (1:n)', (1:n)', ones(n,1))+sparse((2:n)',(1:n-1)',...
    2*ones(n-1,1), n, n)+sparse((3:n)',(1:n-2)',3*ones(n-2,1), n, n);
```

(ii) **a)** Let $\mathbf{A}$ be a matrix whose non-zero entries form a pattern shaped like the letter Z, and denote $\mathbf{B} = \mathbf{A} \cdot \mathbf{A}$. For an arbitrary element of $\mathbf{B}$ by the definition of matrix-matrix multiplication we have

$$b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Due to the definition of $\mathbf{A}$ we have that $a_{ik} \neq 0$ only if either $i = 1$, $i = n$ or $k = n+1-i$. Cases when $i = 1$ and $i = n$ refer to the first and the last row of $\mathbf{B}$. In the last case, $k = n + 1 - i$, we have

$$b_{ij} = a_{i,n+1-i} a_{n+1-i,j}$$

Now, $a_{n+1-i,j}$ is non-zero only if $j = n + 1 - (n + 1 - i) = i$. Therefore, b_{ij} is non-zero only

- $i = 1$,
- $i = n$,
- $j = i$.

In other words, $\mathbf{B}$ is sparse, and it has non-zero entries forming a pattern shaped like the reflected letter Z, that is, shaped like Σ

b) Let $\mathbf{A}$ be a matrix whose non-zero entries form a pattern shaped like the letter X, and denote $\mathbf{B} = \mathbf{AA}$. For an arbitrary element of $\mathbf{B}$ by the definition of matrix-matrix multiplication we have

$$b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Due to the definition of $\mathbf{A}$ we have that $a_{ik} \neq 0$ only if $k = i$ or $k = n+1-i$. Therefore

$$b_{ij} = a_{ii} a_{ij} + a_{i,n+1-i} a_{n+1-i,j}.$$

Furthermore, we have that a_{ij} and $a_{n+1-i,j}$ are non-zero only if $j = 1$ or $j = n + i - 1$. Therefore, b_{ij} is non-zero only

- $j = i$,
- $j = n + 1 - i$.

In other words, $\mathbf{B}$ is sparse and it has non-zero entries forming a pattern shaped like the letter X.

c) Let $\mathbf{A}$ be a three-band matrix as described in the wording of the problem, and denote $\mathbf{B} = \mathbf{AA}$. For an arbitrary element of $\mathbf{B}$ by the definition of matrix-matrix multiplication we have

$$b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

By the definition of $\mathbf{A}$ we have that, for $i \geq 3$, entry a_{ik} is non-zero only if $k = i, i+1$ or $k = i+2$. Hence, we have

$$b_{ij} = a_{ii}a_{ij} + a_{i,i+1}a_{i+1,j} + a_{i,i+2}a_{i+2,j}.$$

Applying the same logic again to $a_{ij}, a_{i+1,j}$ and $a_{i+2,j}$ we have that b_{ij} is non-zero only if $j = i, i+1, i+2, i+3$ or $j = i+4$. That is because $a_{ij} \neq 0$ only if $j = i, i+1$ or $j = i+2$, $a_{i+1,j} \neq 0$ only if $j = i+1, i+2$ or $j = i+3$, and finally, $a_{i+2,j} \neq 0$ only if $j = i+2, i+3$ or $j = i+4$. Therefore, $\mathbf{B}$ is a sparse, five-band matrix, with non-zero entries on the main diagonal and the four sub-diagonals below the main diagonal.

(iii)

```
%% Problem 6.(iii).a)
% INPUT
% x - a vector
% OUTPUT
% y - result of multiplying x by a Z-shaped sparse matrix.

function y = MultiplyZShaped( x )
n = length(x);
y = zeros(size(x)); % Initialising the vector

% First and last entry of y abide to a different rule than other entries
y(1) = dot(ones(size(x)), x);
y(n) = n*dot(ones(size(x)), x);

% The remaining entries
y(2:n-1)=(2:(n-1)).'.* x(n-1:-1:2);
end
```

```
%% Problem 6.(iii).b)
% INPUT
% x - a vector
% OUTPUT
% y - result of multiplying x by a X-shaped sparse matrix.

function y = MultiplyXShaped( x )

y = zeros(size(x)); % Initialising vector y

for i = 1 : length(x)
    y(i) = 2*(x(i)+x(length(x)-i+1));
end

% Same as when constructing X, we deal with the case when x has an odd
% number of entries separately.
if mod(length(x), 2)
    y( (length(x)+1)/2) = y( (length(x)+1)/2) /2;
end
```

```
%% Problem 6.(iii).c)
% INPUT
% x - a vector
% OUTPUT
```

Bitte wenden!

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% y - result of multiplying x by a three band sparse matrix.

function y = MultiplyThreeBand( x)

y = zeros(size(x)); % Initialising y

% First and second entry of y adhere to a different rule than other entries
y(1) = x(1);
y(2) = 2*x(1)+x(2);

% Computing the remaining entries
for i = 3 : length(y)
    y(i) = 3*x(i-2)+2*x(i-1)+x(i);
end
```