Solutions - Problem Sheet 5

1. If we were to combine x_1 parts of M_1, x_2 parts of alloy M_2 and x_3 parts of alloy M_3, the resulting alloy would have the following amounts of copper silver and gold

$$\begin{aligned} \text{Copper} &: \quad 20x_1 + 70x_2 + 50x_3 \\ \text{Silver} &: \quad 60x_1 + 10x_2 + 50x_3 \\ \text{Gold} &: \quad 20x_1 + 20x_2 + 0 \cdot x_3 \end{aligned}$$

In order to obtain the correct values for x_1, x_2 and x_3, we can set the right hand side of those equations to be 40, 50 and 10, in that order. Therefore, if we denote $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, finding x_1, x_2 and x_3 is reduced to finding the solution of the system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 20 & 70 & 50 \\ 60 & 10 & 50 \\ 20 & 20 & 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 40 \\ 50 \\ 10 \end{pmatrix}.$$

To solve the governing system, we use Gaussian eliminations.

$$\left(\begin{array}{ccc|c} 20 & 70 & 50 & 40 \\ 60 & 10 & 50 & 50 \\ 20 & 20 & 0 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3.5 & 2.5 & 2 \\ 60 & 10 & 50 & 50 \\ 20 & 20 & 0 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3.5 & 2.5 & 2 \\ 0 & -20 & -10 & -7 \\ 0 & -5 & -5 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0.75 & 0.775 \\ 0 & 1 & 0.5 & 0.35 \\ 0 & 0 & -2.5 & -1.25 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 1 & 0.5 \end{array}\right)$$

Therefore, to get the alloy with desired ratios of copper, silver and gold, the composition of our mix should have 40% of alloy M_1, 10% of alloy M_2 and 50% of alloy M_3.

2. a) 1. We have

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & 5 & -1 \\ -3 & -8 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the rank of A is 3.

2. The system has a solution if the rank of the augmented matrix is equal to the rank of the non-augmented matrix. We have

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 2 & 5 & -1 & 2 \\ -3 & -8 & 4 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right).$$

Thus, the system has a solution, and that solution is $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$.

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b) 1. We have

$$\begin{pmatrix}1 & -1 & 2\\ 3 & 2 & -1\\ k & 5 & -4\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 2\\ 0 & 5 & -7\\ 0 & 5+k & -4-2k\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0.6\\ 0 & 1 & -1.4\\ 0 & 0 & \frac{15-3k}{5}\end{pmatrix}$$

Therefore, we have two possibilities, firstly if $k \neq 5$ then by the preceeding we have

$$\begin{pmatrix}1 & -1 & 2\\ 3 & 2 & -1\\ k & 5 & -4\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and the rank is 3. Otherwise, if $k = 5$ the rank is 2.

2. We have

$$\left(\begin{array}{ccc|c}1 & -1 & 2 & 3\\ 3 & 2 & -1 & 1\\ k & 5 & -4 & 1\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & -1 & 2 & 3\\ 0 & 5 & -7 & -8\\ 0 & 5+k & -4-2k & 1-3k\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 0.6 & 1.4\\ 0 & 1 & -1.4 & -1.6\\ 0 & 0 & \frac{15-3k}{5} & \frac{45-7k}{5}\end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c}1 & 0 & 0 & \frac{6}{3k-15}\\ 0 & 1 & 0 & \frac{5k-39}{3k-15}\\ 0 & 0 & 1 & \frac{7k-45}{3k-15}\end{array}\right).$$

Hence, the solution exists only if $k \neq 5$, and in that case the solution is $\mathbf{x} = \begin{pmatrix}\frac{6}{3k-15}\\ \frac{5k-39}{3k-15}\\ \frac{7k-45}{3k-15}\end{pmatrix}$.

c) 1. We have

$$\begin{pmatrix}1 & m & 1\\ 1 & 1 & 1\\ m & 1 & m-1\end{pmatrix} \sim \begin{pmatrix}1 & m & 1\\ 0 & 1-m & 0\\ 0 & 1-m^2 & -1\end{pmatrix}.$$

Therefore, if $m = 1$ the Gaussian algorithm will stop, since all the entries in the second column of the second and third row would be zero (so we cannot even swap the rows). Furthermore, if $m = 1$ the entire second row would be equal to $\mathbf{0}$. In conclusion, if $m = 1$ the rank would be 2 since we would have then

$$\begin{pmatrix}1 & m & 1\\ 1 & 1 & 1\\ m & 1 & m-1\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Otherwise, if $m \neq 1$ we have

$$\begin{pmatrix}1 & m & 1\\ 1 & 1 & 1\\ m & 1 & m-1\end{pmatrix} \sim \begin{pmatrix}1 & m & 1\\ 0 & 1-m & 0\\ 0 & 1-m^2 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1-m\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Hence, if $m \neq 1$ the rank is 3.

2. We have

$$\left(\begin{array}{ccc|c}1 & m & 1 & 1\\ 1 & 1 & 1 & m+1\\ m & 1 & m-1 & m\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & m & 1 & 1\\ 0 & 1-m & 0 & m\\ 0 & 1-m^2 & -1 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & m & 1 & 1\\ 0 & 1-m & 0 & m\\ 0 & 1-m^2 & -1 & 0\end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c}1 & 0 & 1 & 1-\frac{m^2}{1-m}\\ 0 & 1 & 0 & \frac{m}{1-m}\\ 0 & 0 & -1 & -m(m+1)\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 0 & \frac{1-2m-m^2+m^3}{1-m}\\ 0 & 1 & 0 & \frac{m}{1-m}\\ 0 & 0 & 1 & m(m+1)\end{array}\right).$$

We see that if $m = 1$ there is no solution. Otherwise the solution is $\mathbf{x} = \begin{pmatrix} \frac{1-2m-m^2-m^3}{1-m} \\ \frac{m}{1-m} \\ m(m+1) \end{pmatrix}$.

3. **a)** We have

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let us solve the system using Gaussian eliminations. We have

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -0.5 & 0 & 0 & 0.5 \\ 0 & 1.5 & -1 & 0 & 0.5 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{4}{3} & -1 & \frac{1}{3} \\ 0 & 0 & -1 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -0.25 & 0.75 \\ 0 & 1 & 0 & -0.5 & 0.5 \\ 0 & 0 & 1 & -0.75 & 0.25 \\ 0 & 0 & 0 & 1.25 & 1.25 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

Therefore, the solution is $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

b)

```
%% Problem 3b
% INPUT
% b - left hand side vector
% OUTPUT
% x - solution of Ax=b, where A is the tridiagonal matrix as described in
% part a

function x = SolverTridiag( b )

n = length(b);

% Creating the matrix
A = 2*diag(ones(n, 1)) - diag(ones(n-1, 1), 1) - diag(ones(n-1, 1), -1);

% Solving the linear system
x = A\b;
```

c) We have

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ n+1 \end{pmatrix}.$$

Multiplying the matrix and the vector on the left hand side yields the following equations

$$\begin{aligned} -x_1 + 2x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ &\cdots \\ -x_{n-2} + 2x_{n-1} - x_n &= 0 \\ -x_{n-1} + 2x_n &= n+1. \end{aligned}$$

The first equation gives us $2x_2 = x_1$. Plugging this expression into the second equation we have

$$x_3 = 2x_2 - x_1 = 3x_1.$$

Applying this algorithm successively to all equations until (but not including) the last one, we have $x_i = ix_1$, for $i = 2, \dots, n$. Using this expression in the last equation we have

$$n + 1 = 2x_n - x_{n-1} = (2n - n + 1)x_1 = (n+1)x_1 \Rightarrow x_1 = 1.$$

Therefore, $x_i = i$, for $i = 1, \dots, n$, and the solution is $\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix}$.

d) Let us first apply Gaussian elimination on this system, in hopes that such an approach will be sensible. We have

$$\left(\begin{array}{ccccc|c} 2 & -1 & & & & b_1 \\ -1 & 2 & -1 & & & b_2 \\ & \ddots & \ddots & \ddots & & \vdots \\ & -1 & 2 & -1 & & b_{n-1} \\ & & -1 & 2 & & b_n \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -\frac{1}{2} & 0 & & & \frac{b_1}{2} \\ 0 & \frac{3}{2} & -1 & & & b_2 + \frac{b_1}{2} \\ & \ddots & \ddots & \ddots & & \vdots \\ & -1 & 2 & -1 & & b_{n-1} \\ & & -1 & 2 & & b_n \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -\frac{1}{2} & 0 & & & \frac{b_1}{2} \\ 0 & 1 & -\frac{2}{3} & 0 & & \frac{2b_2+b_1}{3} \\ 0 & -1 & 2 & -1 & 0 & b_3 \\ & & \ddots & \ddots & \ddots & \\ & & 0 & -1 & 2 & b_n \end{array}\right) \sim \cdots$$

$$\sim \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & & \frac{b_1}{2} \\ 0 & 1 & -\frac{2}{3} & & \frac{2b_2+b_1}{3} \\ & \ddots & \ddots & \ddots & \vdots \\ & 0 & 1 & -\frac{n-1}{n} & \frac{(n-1)b_{n-1}+\cdots+b_1}{n} \\ & 0 & -1 & 2 & b_n \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & & \frac{b_1}{2} \\ 0 & 1 & -\frac{2}{3} & & \frac{2b_2+b_1}{3} \\ & \ddots & \ddots & \ddots & \vdots \\ & & 1 & -\frac{n-1}{n} & \frac{(n-1)b_{n-1}+\cdots+b_1}{n} \\ & & 0 & 1 & \frac{nb_n+\cdots+b_1}{n+1} \end{array}\right)$$

Therefore, we have $x_n = \frac{nb_n+\cdots+b_1}{n+1}$. We can now plug this value into the preceding equation, and thus we would have x_{n-1}. We can continue in this way, once we know x_{i+1}, we can plug it in and thus get x_i, until we have x_1 and we are done.

These considerations can be rigorously formalised. Here is the resulting MATLAB code.

```
%% Problem 3d
% INPUT
% b - right hand side of Ax = b
% OUTPUT
% x - solution of Ax = b

function x = LoopSolverTridiag(b)

n = length(b);

x(n) = dot(b, (1:n)')/(n+1);

for i = n-1:-1:1
    x(i) = x(i+1)*i/(i+1) + dot(b(1:i), (1:i)')/(i+1);
end
```

4. **a)** We have

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 2 & 0 & 2 & -4 \\ 0 & 0 & 3 & 2 & 1 \\ 1 & 1 & 1 & 4 & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 2 & 0 & 2 & -4 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 1 & 1 & 2 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right).$$

Therefore, the solution is $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

b) There will be a unique solution only if the matrix has a full rank. Let us use Gaussian elimination.

$$\begin{pmatrix} d_1 & & & c_1 \\ & \ddots & & \vdots \\ & & d_{n-1} & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{pmatrix} \sim \begin{pmatrix} 1 & & & \frac{c_1}{d_1} \\ & \ddots & & \vdots \\ & & d_{n-1} & c_{n-1} \\ 0 & \cdots & c_{n-1} & c_n - \frac{c_1^2}{d_1} \end{pmatrix} \sim \cdots \sim$$

$$\sim \begin{pmatrix} 1 & & & \frac{c_1}{d_1} \\ & \ddots & & \vdots \\ & & 1 & \frac{c_{n-1}}{d_{n-1}} \\ 0 & \cdots & 0 & c_n - \frac{c_1^2}{d_1} - \cdots - \frac{c_{n-1}}{d_{n-1}} \end{pmatrix}$$

Therefore, there is a unique solution if

$$c_n - \frac{c_1^2}{d_1} - \cdots - \frac{c_{n-1}}{d_{n-1}} \neq 0.$$

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c)

```
%% Problem 4c
% INPUT
% d, c - vectors which define the matrix A
% b - right hand side of Ax = b
% OUTPUT
% x - solution of Ax=b

function x = SpecialSolver(d, c, b)
% We proceed by assuming that the conditions for the existence of a unique
% solution have been met.

n = length(c);

% Building the matrix A
A = sparse(1:n-1, 1:n-1 ,d, n, n) + sparse(n*ones(n,1), 1:n, c, n, n) +...
    +sparse(1:n, n*ones(n, 1), c, n, n);
A(n, n) = c(n);

% Find the solution of Ax=b, using an auxiliary function
x = A\b;
end
```

5. We will use Gaussian eliminations. Therefore, for the treatment of

$$\left(\begin{array}{cc|c} a & b & b_1 \\ c & d & b_2 \end{array}\right).$$

we have to distinguish cases in each step.

1. If $a = 0$, we have to swap first row with the second one, if it is possible to do so.
   (a) If $c = 0$ we have
   $$\left(\begin{array}{cc|c} 0 & b & b_1 \\ 0 & d & b_2 \end{array}\right).$$
   There are 4 possibilities at this point
   - If $d = 0$ and $b = 0$ the solution exists only if $b_1 = b_2 = 0$, and solutions are $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$, i.e., any pair of real numbers.
   - If $d = 0$ and $b \neq 0$ the solution exists only if $b_1 = 0$, and the solution is $x_2 = \frac{b_2}{b}$ and $x_1 \in \mathbb{R}$ is any real number.
   - If $d \neq 0$ and $b = 0$ the solution exists only if $b_2 = 0$, and the solution is $x_2 = \frac{b_1}{d}$ and $x_1 \in \mathbb{R}$ is any real number.
   - Otherwise, if $d \neq 0$ and $b \neq 0$ the solution exists only if $\frac{b_1}{d} = \frac{b_2}{b}$. Then the solution is $x_2 = \frac{b_1}{d}$ and $x_1 \in \mathbb{R}$ is any real number.

   (b) If $c \neq 0$ we have
   $$\left(\begin{array}{cc|c} c & d & b_2 \\ 0 & b & b_1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{d}{c} & \frac{b_2}{c} \\ 0 & b & b_1 \end{array}\right).$$
   Again, we have a couple of cases.
   - If $b = 0$ the solution exists only if $b_1 = 0$, and in that case the solution is $x_1 = \frac{b_2}{c} - \frac{d}{c}x_2$, where x_2 is any real number.

– If $b \neq 0$ we have

$$\left(\begin{array}{cc|c} 1 & \frac{d}{c} & \frac{b_2}{c} \\ 0 & b & b_1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{b_2}{c} - \frac{d}{c}\frac{b_1}{b} \\ 0 & 1 & \frac{b_1}{b} \end{array}\right).$$

2. If $a \neq 0$ we have

$$\left(\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{b_1}{a} \\ 0 & d - \frac{b}{a}c & b_2 - \frac{b_1}{a}c \end{array}\right).$$

(a) If $ad - bc = 0$, the solution exists only if $b_2 a - b_1 c = 0$. The solution in that case is $x_1 = \frac{b_1}{a} - \frac{b}{a}x_2$, where x_2 is any real number.

(b) If $ad - bc \neq 0$ we have

$$\left(\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{b_1}{a} \\ 0 & d - \frac{b}{a}c & b_2 - \frac{b_1}{a}c \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{db_1 - bb_2}{ad-bc} \\ 0 & 1 & \frac{ab_2 - cb_1}{ad-bc} \end{array}\right).$$

6. We will map the basis $\{1, x, x^2, \dots, x^7\}$ of $\mathbb{P}^7$ to the standard basis $\{\mathbf{e}^1, \dots \mathbf{e}^8\}$ of $\mathbb{R}^8$, by the $x^i \to \mathbf{e}^i$, where $\mathbf{e}^i$ has a 1 at the i-th place, and zeros for all other entries.

a) Using the previsouly mentioned mapping, we have

$$p_1(x) \to \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2(x) \to \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_3(x) \to \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

These 3 vectors will be linearly indepenedent if the rank of the matrix which has them as rows, has rank 3, i.e., if the rank is equal to the number of vectors. We have

$$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the rank is 3 and we conclude that the given set of polynomials is indeed linearly independent.

b) We have

$$p_1(x) \to \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2(x) \to \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_3(x) \to \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad p_4(x) \to \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For the linear independence, we have

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the rank is 4 and we conclude that the given set of polynomials is indeed linearly independent.

c) We have

$$p_1(x) \to \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2(x) \to \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_3(x) \to \begin{pmatrix} 4 \\ 0 \\ 4 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_4(x) \to \begin{pmatrix} 2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_5(x) \to \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Let us compute the rank. We have

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 1 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 4 & -3 & 0 & 1 & 0 & 0 \\ 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0.4375 & 0 & 0.1875 & 0 & 0 \\ 0 & 1 & 0 & 0.1875 & 0 & -0.0625 & 0 & 0 \\ 0 & 0 & 1 & -0.1875 & 0 & 0.0625 & 0 & 0 \\ 0 & 0 & 0 & 1.6875 & 0 & -0.5625 & 0 & 0 \\ 0 & 0 & 0 & 0.5625 & 1 & 0.8125 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Therefore, the rank is 5 and we conclude that the given set of polynomials is indeed linearly independent.

7. Since $\mathbf{A}$ is invertible, that is, since $\mathbf{A}^{-1}$ exists, the matrix

$$\mathbf{B} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}$$

is well defined. We have

$$\begin{aligned}(\mathbf{A} + \mathbf{u}\mathbf{v}^T)\left(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}\right) &= \mathbf{A}\mathbf{A}^{-1} - \mathbf{A}\mathbf{A}^{-1}\frac{\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - \mathbf{u}\mathbf{v}^T\frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \\ &= \mathbf{I} - \frac{\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - (\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})\frac{\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \\ &= \mathbf{I} - (1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})\frac{\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} \\ &= \mathbf{I} - \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} = \mathbf{I},\end{aligned}$$

where the second equality followed from the fact that $\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}$ is just a number, that is, a scalar, and we know that for a scalar $a \in \mathbb{R}$ and matrices $\mathbf{C}, \mathbf{D}$ we have

$$\mathbf{C}(a\mathbf{D}) = a(\mathbf{C}\mathbf{D}).$$

In conclusion, $\mathbf{B}$ is indeed the inverse of $\mathbf{A} + \mathbf{u}\mathbf{v}^T$.

8. **a)** We begin by writing vectors in $\mathcal{A}$ as linear combinations of vectors in $\mathcal{B}$. In order to do that we need to solve some linear system, which we will skip here for the sake of brevity.

We have

$$\begin{pmatrix}1\\-1\\2\end{pmatrix} = 1\begin{pmatrix}1\\2\\2\end{pmatrix} + 6\begin{pmatrix}-1\\3\\3\end{pmatrix} - 3\begin{pmatrix}-2\\7\\6\end{pmatrix},$$

$$\begin{pmatrix}2\\3\\7\end{pmatrix} = 2.6\begin{pmatrix}1\\2\\2\end{pmatrix} + 8.6\begin{pmatrix}-1\\3\\3\end{pmatrix} - 4\begin{pmatrix}-2\\7\\6\end{pmatrix},$$

$$\begin{pmatrix}2\\3\\6\end{pmatrix} = 2.4\begin{pmatrix}1\\2\\2\end{pmatrix} + 6.4\begin{pmatrix}-1\\3\\3\end{pmatrix} - 3\begin{pmatrix}-2\\7\\6\end{pmatrix}.$$

Therefore, matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix}1 & 2.6 & 2.4\\6 & 8.6 & 6.4\\-3 & -4 & -3\end{pmatrix}.$$

b) The coordinates of $\mathbf{v}$ in basis $\mathcal{B}$ are given as a multiple of $\mathbf{S}$ and the coordinates of $\mathbf{v}$ in basis $\mathcal{A}$. We have

$$\mathbf{S}\begin{pmatrix}2\\9\\-8\end{pmatrix} = \begin{pmatrix}1 & 2.6 & 2.4\\6 & 8.6 & 6.4\\-3 & -4 & -3\end{pmatrix}\begin{pmatrix}2\\9\\-8\end{pmatrix} = \begin{pmatrix}6.2\\38.2\\-18\end{pmatrix}.$$

In other words,

$$\mathbf{v} = 6.2\begin{pmatrix}1\\2\\2\end{pmatrix} + 38.2\begin{pmatrix}-1\\3\\3\end{pmatrix} - 18\begin{pmatrix}-2\\7\\6\end{pmatrix}.$$