Solutions - Problem Sheet 10

1. We want to find all solutions to the equation

$$\alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_n \mathbf{b}^n = \mathbf{0} \tag{1}$$

where $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Let us apply P to (1). We have

$$\mathsf{P}\left(\sum_{i=1}^{n} \alpha_i \mathbf{b}^i\right) = \sum_{i=1}^{n} \alpha_i \mathsf{P}\mathbf{b}^i = \mathbf{0}. \tag{2}$$

Since $\mathbf{b}^i \in \ker(\mathsf{P})$ for $i = r+1, \ldots, n$ we have

$$\mathsf{P}\mathbf{b}^i = \mathbf{0}, \quad \text{for } i = r+1, \ldots, n.$$

Furthermore, since $\mathbf{b}^i \in \mathrm{im}(\mathsf{P})$ for $i = 1, \ldots, r$ we have

$$\mathsf{P}\mathbf{b}^i = \mathbf{b}^i, \quad \text{for } i = 1, \ldots, r.$$

Combining those two facts we have that (2) is equivalent to

$$\alpha_1 \mathbf{b}^1 + \cdots + \alpha_r \mathbf{b}^r = \mathbf{0}$$

Since $\{\mathbf{b}^1, \ldots, \mathbf{b}^r\}$ is a basis of $\mathrm{im}(\mathsf{P})$ it is also a linearly independent set. Therefore $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$. This reduces (1) to

$$\alpha_{r+1} \mathbf{b}^{r+1} + \alpha_{r+2} \mathbf{b}^{r+2} + \cdots + \alpha_n \mathbf{b}^n = \mathbf{0}.$$

Again, since $\{\mathbf{b}^{r+1}, \ldots, \mathbf{b}^n\}$ is a basis for $\ker(\mathsf{P})$ it is also a linearly independent set. Thus $\alpha_{r+1} = \ldots = \alpha_n = 0$. In conclusion, we have $\alpha_i = 0$ for all $i = 1, \ldots, n$ which means that $\{\mathbf{b}^1, \ldots, \mathbf{b}^n\}$ is a linearly independent set.

Therefore, it is a set of n linearly independent vectors in a vector space of dimension n, so it is a basis for V.

2. a) Let $\mathbf{x}, \mathbf{y} \in U$ and define $\mathbf{z} = \mathbf{x} + \mathbf{y}$. We have

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{z} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{x} + \mathbf{y} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{x} \right\rangle + \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{y} \right\rangle = 0.$$

Therefore, $\mathbf{x} + \mathbf{y} \in U$.

Take $\alpha \in \mathbb{R}$ and $\mathbf{x} \in U$ and define $\mathbf{z} = \alpha\mathbf{x}$. We have

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{z} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \alpha\mathbf{x} \right\rangle = \alpha \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{x} \right\rangle = 0.$$

Therefore, $\alpha\mathbf{x} \in U$ and we have that U is indeed a subspace of V.

b) Take an arbitrary $\mathbf{x} \in U$. Then

$$0 = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{x} \right\rangle = x_1 + 2x_2 + 3x_3 \Rightarrow x_1 = -2x_2 - 3x_3.$$

This means that

$$\begin{aligned} U = \{\mathbf{x} \in \mathbb{R}^3 : x_1 = -2x_2 - 3x_3\} &= \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \begin{pmatrix} -2s - 3t \\ s \\ t \end{pmatrix}, s, t \in \mathbb{R} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, s, t \in \mathbb{R} \right\}, \end{aligned}$$

that is,

$$U = \mathsf{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since the set of vectors $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is clearly linearly independent it is also a basis for U.

c) We will use Theorem 4.5.E. The set $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for U while $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for W. Therefore,

$$\mathcal{B}_V = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$ by Problem 1**c)**. Theorem 4.5.E now tells us that with respect to $\mathcal{B}_V$, the matrix representation of our projection is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

d) We will determine the matrix representation of the projection P in Cartesian cooridinates by computing the change of basis. The first step is to write the elements of one basis as the linear combination of the elements of the other basis. We have

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= -\frac{1}{6} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \frac{2}{3} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Siehe nächstes Blatt!

which gives

$$\mathbf{S} = \begin{pmatrix} -\frac{1}{6} & \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

Since the other basis is the standard Cartesian basis, we have

$$\mathbf{R} = \begin{pmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Now we have

$$\mathbf{P} = \mathbf{RAS} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

3. a) The matrix $\mathbf{B}^\top\mathbf{B}$ is symmetric, since

$$(\mathbf{B}^\top\mathbf{B})^\top = \mathbf{B}^\top(\mathbf{B}^\top)^\top = \mathbf{B}^\top\mathbf{B}.$$

Therefore, the upper triangular part describes the whole matrix. Since an $n \times n$ matrix has $\frac{n(n+1)}{2}$ entries in its upper triangular part, we have that there are 6 defining equations for our a, b, c, d, e and f, because in our case $n = 3$.

b) Computing $\mathbf{B}^\top\mathbf{B} = \mathbf{I}$ we have

(i) $\frac{1}{3} + \frac{1}{3} + e^2 = 1$,
(ii) $\frac{1}{\sqrt{3}}a - \frac{1}{\sqrt{3}}c = 0$,
(iii) $\frac{1}{\sqrt{3}}b + \frac{1}{\sqrt{3}}d - fe = 0$,
(iv) $a^2 + c^2 = 1$,
(v) $ab - cd = 0$,
(vi) $b^2 + d^2 + f^2 = 1$.

From (i) we have $e = \frac{1}{\sqrt{3}}$; from (ii) we have $a = c$. Plugging this into (iv) we have $a = c = \frac{1}{\sqrt{2}}$. From (v) we have $b = d$ and (iii) then gives us $f = 2b$. Finally, plugging both of those expressions in (vi) yields $b = d = \frac{1}{\sqrt{6}}$ and $f = \frac{2}{\sqrt{6}}$.

4. Let us denote the elements of the Cartesian basis as follows

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Reflection about the plane $\{x_1 = x_2\}$ is invariant to the elements of that plane. Thus, $\mathsf{F}_1(\mathbf{e}^3) = \mathbf{e}^3$. Also, we easily see $\mathsf{F}_1(\mathbf{e}^1) = \mathbf{e}^2$ and $\mathsf{F}_1(\mathbf{e}^2) = \mathbf{e}^1$ and the corresponding matrix representation is

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the map F_2 the Section 4.6 of the lectures gives us (with $\phi = \pi/4$)

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

An analogous equation holds for F_3. Rotating around the x_2 axis leaves the vectors which are on that axis unchanged, hence $\mathsf{F}_3(\mathbf{e}^2) = \mathbf{e}^2$. On the other hand we have

$$\mathsf{F}_3(\mathbf{e}^1) = \begin{pmatrix} \cos(\pi/6) \\ 0 \\ \sin(\pi/6) \end{pmatrix} \quad \text{and} \quad \mathsf{F}_3(\mathbf{e}^3) = \begin{pmatrix} -\sin(\pi/6) \\ 0 \\ \cos(\pi/6) \end{pmatrix}.$$

Therefore

$$\mathbf{A}_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

b) Matrix representation of a linear map that is given as a composition of two other linear maps is just a product of matrix representations of those two maps. Therefore, the matrix representation of $\mathsf{F}_2 \circ \mathsf{F}_1$ is

$$\mathbf{A}_2\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Similarly, for $\mathsf{F}_3 \circ \mathsf{F}_2$ we have

$$\mathbf{A}_3\mathbf{A}_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}.$$

5. As in Problem 3, we compute $\mathbf{A}^\top\mathbf{A}$ and equate it to $\mathbf{I}$. By doing so we obtain the following equations

(i) $\frac{1}{2} + r^2 = 1$,

(ii) $\frac{s}{\sqrt{2}} + \frac{r}{\sqrt{2}} = 0$,

(iii) $\frac{t}{\sqrt{2}} = 0$,

(iv) $\frac{1}{2} + s^2 = 1$,

(v) $ts = 0$,

(vi) $1 + t^2 = 1$.

From (vi) and (iii) we have $t = 0$ while (ii) gives $s = -r$. Then (i) gives $r = \pm\frac{1}{\sqrt{2}}$ and $s = \mp\frac{1}{\sqrt{2}}$. Hence, we either have $r = \frac{1}{\sqrt{2}}$, $s = -\frac{1}{\sqrt{2}}$, $t = 0$, or $r = -\frac{1}{\sqrt{2}}$, $s = \frac{1}{\sqrt{2}}$, $t = 0$.

6. **a)** It is sufficient to show that the corresponding matrix representations are orthogonal matrices. We have

$$\begin{aligned} \mathbf{A}^\top\mathbf{A} &= (\mathbf{I}_2 - 2\mathbf{u}\mathbf{u}^\top)^\top(\mathbf{I}_2 - 2\mathbf{u}\mathbf{u}^\top) = (\mathbf{I}_2 - 2\mathbf{u}\mathbf{u}^\top)(\mathbf{I}_2 - 2\mathbf{u}\mathbf{u}^\top) = \mathbf{I}_2 - 4\mathbf{u}\mathbf{u}^\top + 4\mathbf{u}(\mathbf{u}^\top\mathbf{u})\mathbf{u}^\top \\ &= \mathbf{I}_2 - 4\mathbf{u}\mathbf{u}^\top + 4\mathbf{u}\|\mathbf{u}\|\mathbf{u}^\top = \mathbf{I}_2. \end{aligned}$$

Analogous equations hold for $\mathbf{B}$. Hence, F and G are isometries.

Siehe nächstes Blatt!

b) It is sufficient to find $\mathbf{w} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{w} = \mathbf{w}$. We have

$$\mathbf{A}\mathbf{w} = (\mathbf{I}_2 - 2\mathbf{u}\mathbf{u}^\top)\mathbf{w} = \mathbf{w} - 2\mathbf{u}(\mathbf{u}^\top\mathbf{w})$$

Therefore, if we take $\mathbf{w}$ to be a vector such that $\mathbf{u}^\top\mathbf{w} = 0$ then $\mathbf{A}\mathbf{w} = \mathbf{w}$ follows. Such non-zero vectors do exist, for example, for $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ if we take $\mathbf{w} = \begin{pmatrix} -y \\ x \end{pmatrix}$ then $\mathbf{u}^\top\mathbf{w} = 0$ and also $\mathbf{w} \neq \mathbf{0}$ since $\|\mathbf{w}\| = \|\mathbf{u}\| = 1$. Furthermore, for $\alpha \in \mathbb{R}$ we have $\mathbf{A}(\alpha\mathbf{w}) = \alpha\mathbf{A}\mathbf{w} = \alpha\mathbf{w}$. Therefore, the space

$$U = \operatorname{Span}\left\{\begin{pmatrix} -y \\ x \end{pmatrix}\right\},$$

where $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, is a one-dimensional subspace of $\mathbb{R}^2$ whose elements are mapped to themselves, with respect to F. By analogy the same follows for G.

c) We can notice that $\mathbf{A}\mathbf{u} = -\mathbf{u}$. Furthermore, we know that for all vectors orthogonal to $\mathbf{u}$, that is, all vectors $\mathbf{w}$ such that $\mathbf{w}\top\mathbf{u} = 0$, $\mathbf{A}\mathbf{w} = \mathbf{w}$ holds. Hence, F and G are reflection about a line (containing the origin) of direction $\mathbf{w}$

d) Since $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ both $\mathbf{u}$ and $\mathbf{v}$ lie on the unit circle. Therefore, there exist $\varphi, \vartheta \in [0, 2\pi]$ such that

$$\mathbf{u} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix}.$$

We compute

$$\mathbf{u}\mathbf{u}^\top = \begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi \end{pmatrix}.$$

Plugging this into the expression for $\mathbf{A}$, and using trigonometric identities, we have

$$\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^\top = \begin{pmatrix} -\cos(2\varphi) & -\sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi) \end{pmatrix}.$$

An analogous expression holds for $\mathbf{B}$, that is

$$\mathbf{B} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^\top = \begin{pmatrix} -\cos(2\vartheta) & -\sin(2\vartheta) \\ -\sin(2\vartheta) & \cos(2\vartheta) \end{pmatrix}.$$

The matrix representation of the composition of those two maps is then just a product of their corresponding matrix representations. Hence, we have

$$\begin{aligned}\mathbf{B}\mathbf{A} &= \begin{pmatrix} \cos(2\vartheta)\cos(2\varphi) + \sin(2\vartheta)\sin(2\varphi) & \cos(2\vartheta)\sin(2\varphi) - \sin(2\vartheta)\cos(2\varphi) \\ \cos(2\varphi)\sin(2\vartheta) - \sin(2\varphi)\cos(2\vartheta) & \cos(2\vartheta)\cos(2\varphi) + \sin(2\vartheta)\sin(2\varphi) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\vartheta - 2\varphi) & -\sin(2\vartheta - 2\varphi) \\ \sin(2\vartheta - 2\varphi) & \cos(2\vartheta - 2\varphi) \end{pmatrix}\end{aligned}$$

which is a rotation matrix through an angle $2(\vartheta - \varphi)$.

e) Since both F and G are reflections the resulting action is the reflection across two lines (in $\mathbb{R}^2$) which is exactly a rotation for double the angle between those two lines.

7. Denote the vectors of the Cartesian basis of $\mathbb{R}^3$ by

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

a) We have

$$\|\mathbf{u}\|^2 = \cos^2\phi\cos^2\vartheta + \sin^2\phi\cos^2\vartheta + \sin^2\vartheta = \left(\cos^2\phi + \sin^2\phi\right)\cos^2\vartheta + \sin^2\vartheta = \cos^2\vartheta + \sin^2\vartheta = 1.$$

b) We want to compute $\mathsf{S}_{\phi,\vartheta}\mathbf{e}^i$ for $i = 1, 2, 3$. Since $\mathbf{e}^1$ and $\mathbf{e}^2$ already lie in the plane $x_3 = 0$, we have that the first two components of their projections onto that plane are exactly their first two entries. In other words

$$\mathsf{S}_{\phi,\vartheta}(\mathbf{e}^1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathsf{S}_{\phi,\vartheta}(\mathbf{e}^2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In order to compute $\mathsf{S}_{\phi,\vartheta}(\mathbf{e}^3)$ let us write $\mathbf{e}^3$ as a linear combination of $\mathbf{e}^1, \mathbf{e}^2$ and $\mathbf{u}$. This can be done since the set $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{u}\}$ is a basis for $\mathbb{R}^3$. We have

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{\cos\phi\cos\vartheta}{\sin\vartheta}\mathbf{e}^1 - \frac{\sin\phi\cos\vartheta}{\sin\vartheta}\mathbf{e}^2 + \frac{1}{\sin\vartheta}\mathbf{u}.$$

Now, the projection of $\mathbf{e}^3$ onto the plane $\{x_3 = 0\}$ is the projection onto $\operatorname{Span}\{\mathbf{e}^1, \mathbf{e}^2\}$, thus it simply means removing its component in the $\mathbf{u}$ direction. Therefore,

$$\mathsf{S}_{\phi,\vartheta}(\mathbf{e}^3) = \begin{pmatrix} -\frac{\cos\phi\cos\vartheta}{\sin\vartheta} \\ -\frac{\sin\phi\cos\vartheta}{\sin\vartheta} \end{pmatrix}$$

and the matrix representation of $\mathsf{S}_{\phi,\vartheta}$ with respect to Cartesian bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -\frac{\cos\phi\cos\vartheta}{\sin\vartheta} \\ 0 & 1 & -\frac{\sin\phi\cos\vartheta}{\sin\vartheta} \end{pmatrix}.$$

d) We have seen in the lectures that a linear mapping maps lines to lines. Hence, `plot` is sufficient since it only plots lines between given points.

e)

```
%
function plottetshadow(A, phi, theta)

% Creating the matrix representation of our map
P = [1 0 -cos(phi)*cos(theta)/sin(theta);...
     0 1 -sin(phi)*cos(theta)/sin(theta)];

points = zeros(2,4);
% Computing the projected points
for i = 1 : 4
    points(:, i) = P*A(:,i);
end

% Plotting
figure(1)
hold on
for i = nchoosek(1:4, 2)'
    plot([points(1,i(1)) points(1,i(2))], ...
        [points(2,i(1)) points(2,i(2))], 'k-', 'LineWidth', 2);
end

% Making the plot a bit nicer
```

Siehe nächstes Blatt!

```
xlim([min(points(1,:))-1, max(points(1,:))+1]);
ylim([min(points(2,:))-1, max(points(2,:))+1]);

% Alternative way to plot this would be to consecutively plot each of the
% six edges of the projetion by writing its own plot function. Method used
% here achives that same effect with a for loop.
```

8. A tetrahedron will cast a triangular shadow if one if its edges is parallel to the **u**. That is because if one if its edges is parallel to **u** then the projection will map the vertices which define that edge onto a single point, rather than two distinct points.