Problem Sheet 1

1. Determine whether each of the following statements are true for all $x_1, \ldots, x_n,\ y_1, \ldots, y_n \in \mathbb{R}$ and $a, b \in \mathbb{R}$.

- ○ $\sum_{i=1}^n (x_i + y_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i$
- ○ $\sum_{i=1}^n x_i = \sum_{k=1}^n x_k = \sum_{k=1}^n x_{n+1-k}$
- ○ $\sum_{i=1}^n (ax_i + b) = a\big(\sum_{i=1}^n x_i\big) + b$
- ○ $\sum_{i=1}^n (x_i \cdot y_i) = \big(\sum_{i=1}^n x_i\big) \cdot \big(\sum_{i=1}^n y_i\big)$
- ○ $\sum_{i=1}^n (x_i - \frac{1}{n}\sum_{j=1}^n x_j) = 0$
- ○ $\sum_{i=1}^n \sum_{j=1}^n x_i \cdot y_j = \big(\sum_{i=1}^n x_i\big) \cdot \big(\sum_{j=1}^n y_j\big)$
- ○ $(a-1)\big(\sum_{i=0}^n a^i\big) = a^n - 1$

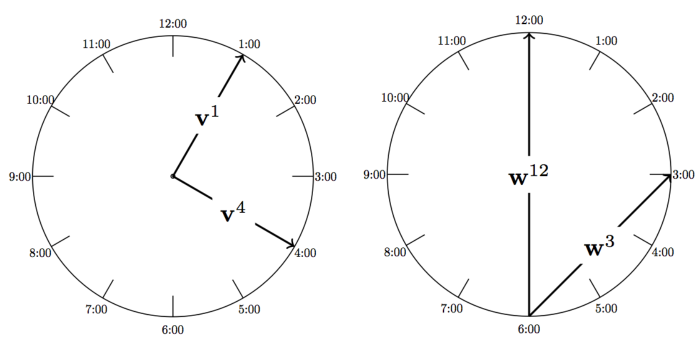
2. In the following, we represent a clock by a unit circle (a circle of radius 1 with a centre at the origin $(0,0)$).

(a) What is the sum $\mathbf{s}$ of the twelve vectors $\mathbf{v}^1, \ldots, \mathbf{v}^{12}$ that go from the centre of a clock to the hours 1:00, 2:00, …, 12:00?

(b) If the vector $\mathbf{v}^4$ (pointing to 4:00) is removed, find the sum of the eleven remaining vectors.

(c) Assume that the vector $\mathbf{v}^1$ (pointing to 1:00) is halved. Add this new vector to the other eleven vectors $\mathbf{v}^2, \ldots, \mathbf{v}^{12}$.

(d) Suppose that the centre of the circle is now at $(0,1)$. Thus, our twelve vectors $\mathbf{w}^1, \ldots, \mathbf{w}^{12}$ start from 6:00 at the bottom. Add the new twelve vectors.



(a) For problems 2.(a)-(c) (b) For problem 2.(d)

Bitte wenden!

3. For the following two problems, let us recall the axioms that vector addition and scalar multiplication defined over a set V must adhere to, for all vectors $\mathbf{v}, \mathbf{w} \in V$ and scalars $\alpha, \beta \in \mathbb{R}$, so that $(V, +, \cdot)$ is a vector space

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
2. $\mathbf{v} + (\mathbf{w} + \mathbf{u}) = (\mathbf{v} + \mathbf{w}) + \mathbf{u}$.
3. There is a unique zero vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
4. For each $\mathbf{v}$ there exists a unique vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. $1 \cdot \mathbf{v} = \mathbf{v}$.
6. $(\alpha \cdot \beta) \cdot \mathbf{v} = \alpha \cdot (\beta \cdot \mathbf{v})$.
7. $\alpha \cdot (\mathbf{v} + \mathbf{w}) = \alpha \cdot \mathbf{v} + \alpha \cdot \mathbf{w}$.
8. $(\alpha + \beta) \cdot \mathbf{v} = \alpha \cdot \mathbf{v} + \beta \cdot \mathbf{v}$.

(a) Suppose that the addition rule in $\mathbb{R}^2$ is defined to be

$$\mathbf{v} \oplus \mathbf{w} = (v_1, v_2) \oplus (w_1, w_2) := (v_1 + w_2, v_2 + w_1).$$

With the standard scalar-multiplication rule

$$\alpha \cdot \mathbf{v} = (\alpha \cdot v_1, \alpha \cdot v_2).$$

Show that this is not a vector space, that is, which of the axioms of a vector space hold, and which ones fail?

(b) Suppose the scalar multiplication $\odot$ is defined to be $\alpha \odot \mathbf{v} = (\alpha \cdot v_1, 0)$ instead of $(\alpha v_1, \alpha v_2)$. With the standard addition in $\mathbb{R}^2$

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)^\top,$$

are all the axioms satisfied for $(\mathbb{R}^2, +, \odot)$ to be a vector space?

4. Take the set of all continuous functions $C(\mathbb{R})$.

(a) Consider $(C(\mathbb{R}), +, \odot)$. Which rule is broken if multiplying $f \in C(\mathbb{R})$ by a scalar $\alpha \in \mathbb{R}$ is defined as

$$(\alpha \odot f)(x) := f(\alpha \cdot x), \text{ for all } x \in \mathbb{R}.$$

while we keep the standard addition rule $(f + g)(x) := f(x) + g(x)$, for all $x \in \mathbb{R}$?

(b) If the sum of "vectors" $f(x)$ and $g(x)$ is defined as

$$(f \oplus g)(x) := f(g(x)) \text{ for all } x \in \mathbb{R},$$

then the "zero vector" is $e(x) = x$. Keep the standard scalar multiplication $(\alpha \cdot f)(x) := \alpha \cdot f(x)$ and consider $(C(\mathbb{R}), \oplus, \cdot)$. Which condtions are broken?

5. Which of the following subsets of $\mathbb{R}^3$ are also subspaces of $(\mathbb{R}^2, +, \cdot)$,with the standard pointwise addition and scalar multiplication ?

(a) The plane of vectors $\{(v_1, v_2, v_3)^\top \in \mathbb{R}^3 : v_1 = v_2\}$.

(b) The plane of vectors with $\{(v_1, v_2, v_3)^\top \in \mathbb{R}^3 : v_1 = 1\}$.

Siehe nächstes Blatt!

(c) The vectors with $\{(v_1, v_2, v_3)^\top \in \mathbb{R}^3 | \; v_1 \cdot v_2 \cdot v_3 = 0\}$.

(d) All vectors that satisfy $\{(v_1, v_2, v_3)^\top \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0\}$.

(e) All vectors with $\{(v_1, v_2, v_3)^\top \in \mathbb{R}^3 : v_1 \leq v_2 \leq v_3\}$.

(f) All linear combinations of $\mathbf{v} = (1, 4, 0)^\top$ and $\mathbf{w} = (2, 2, 3)^\top$.

For the following, let us recall that $C(\mathbb{R})$ represents the set of all continuous functions on $\mathbb{R}$, while $\mathcal{P}_n$ is the set of all polynomials of degree less (or equal) than n, both of which we endow with standard pointwise addition and scalar multiplication. That is, we will consider $(C(\mathbb{R}), +, \cdot)$ and $(\mathcal{P}_n, +, \cdot)$.

(g) Is $\{f(x) \in C(\mathbb{R}) : \int_0^1 f(x)dx = 0\}$ a subspace of $(C(\mathbb{R}), +, \cdot)$?

(h) Is $\{p(x) \in \mathcal{P}_2 : p(0) = 1\}$ a subspace of $(\mathcal{P}_2, +, \cdot)$?

(i) Is $\{p(x) \in \mathcal{P}_7 : p(0) = 2p'(0)\}$ a subspace of $(\mathcal{P}_7, +, \cdot)$?