Problem Sheet 3

1. Let $(V, +, \cdot)$ be a vector space with an inner product $\langle \cdot, \cdot \rangle$, and for $m \in \mathbb{N}$ let $\mathbf{a}^1, \ldots, \mathbf{a}^m$ be non-zero vectors in V, that is, such that $\mathbf{a}^j \neq \mathbf{0}$ for all $j = 1, \ldots, m$.

(a) Let us assume that for some $\mathbf{v} \in V$ we have $\langle \mathbf{v}, \mathbf{a}^j \rangle = 0$ for all $j = 1, \ldots, m$. Show that such a $\mathbf{v}$ is orthogonal to $\mathsf{Span}\{\mathbf{a}^1, \ldots, \mathbf{a}^m\}$, that is, show that $\mathbf{v}$ is orthogonal to all vectors which can be written as a linear combination of vectors $\mathbf{a}^1, \ldots, \mathbf{a}^m$.

(b) Let $V = \mathbb{R}^3$, with default vector addition, scalar multiplication and inner product. For $m = 2$ take $\mathbf{a}^1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, $\mathbf{a}^2 = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Compute a vector $\mathbf{w}$, where

$$\mathbf{w} = \frac{\langle \mathbf{v}, \mathbf{a}^1 \rangle}{\langle \mathbf{a}^1, \mathbf{a}^1 \rangle} \mathbf{a}^1 + \frac{\langle \mathbf{v}, \mathbf{a}^2 \rangle}{\langle \mathbf{a}^2, \mathbf{a}^2 \rangle} \mathbf{a}^2,$$

verify that $\langle \mathbf{a}^1, \mathbf{a}^2 \rangle = 0$ and show that $\mathbf{v} - \mathbf{w}$ is orthogonal to $\mathsf{Span}\{\mathbf{a}^1, \mathbf{a}^2\}$.

(c) For an arbitrary vector $\mathbf{v} \in V$ we define a vector

$$\mathbf{w} :\, = \sum_{j=1}^{m} \frac{\langle \mathbf{v}, \mathbf{a}^j \rangle}{\langle \mathbf{a}^j, \mathbf{a}^j \rangle} \mathbf{a}^j. \tag{1}$$

Show that if all $\mathbf{a}^j$'s are pairwise orthogonal, that is, if we have

$$\langle \mathbf{a}^l, \mathbf{a}^j \rangle = 0 \quad \text{for all} \quad l, j \in \{1, \ldots, m\} \quad \text{such that} \quad l \neq j,$$

then the vector $\mathbf{v} - \mathbf{w}$ is orthogonal to $\mathsf{Span}\{\mathbf{a}^1, \ldots, \mathbf{a}^m\}$.

(d) Let $V = \mathbb{P}_2$, the set of polynomials of degree less than or equal to 2, with standard operations

$$(\mathbf{p} + \mathbf{q})(x) = \mathbf{p}(x) + \mathbf{q}(x) \quad \text{and} \quad (\alpha \mathbf{p})(x) = \alpha \mathbf{p}(x) \quad \text{for all} \quad \mathbf{p}, \mathbf{q} \in \mathbf{P}_2.$$

Let us also define an inner product on $\mathbb{P}_2$ by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 \mathbf{p}(x) \mathbf{q}(x) dx \quad \text{for} \quad \mathbf{p}, \mathbf{q} \in \mathbb{P}_2.$$

Show that, if we take $\mathbf{a}^1, \mathbf{a}^2, \mathbf{v} \in \mathbb{P}_2$ as

$$\begin{aligned} \mathbf{a}^1(x) &= 1, \\ \mathbf{a}^2(x) &= 1 - 2x, \qquad x \in \mathbb{R} \\ \mathbf{v}(x) &= x^2, \end{aligned}$$

then $\mathbf{a}^1$ and $\mathbf{a}^2$ are orthogonal vectors in $\mathbb{P}_2$. Compute *the polynomial* $\mathbf{w}$ using formula (1).

2. (a) Write a MATLAB function

```
function aReflect = Problem2a(a, p, q)
```

that takes as arguments a point $\mathbf{a} \in \mathbb{R}^2$ (a column vector) and column vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$, $\mathbf{q} \neq \mathbf{0}$, which determine a line $\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{p} + \tau\mathbf{q},\ \tau \in \mathbb{R}\}$. The function is supposed to return the reflection of $\mathbf{a}$ across $\mathcal{G}$. Additionaly, plot $\mathbf{a}$, $\mathcal{G}$ and the reflection of $\mathbf{a}$.

Hint: A template for the function `Problem2a` is available online and can be downloaded from course website.

(b) Write a MATLAB function

```
function [s, t] = Problem2b(p1, q1, p2, q2)
```

which for two lines $\mathcal{G}_1 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{p}^1 + \tau\mathbf{q}^1,\ \tau \in \mathbb{R}\}$ and $\mathcal{G}_2 = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{p}^2 + \tau\mathbf{q}^2,\ \tau \in \mathbb{R}\}$, given by column vectors $\mathbf{p}^1, \mathbf{q}^1 \in \mathbb{R}^2$, $\mathbf{q}^1 \neq \mathbf{0}$ and $\mathbf{p}^2, \mathbf{q}^2 \in \mathbb{R}^2$, $\mathbf{q}^2 \neq \mathbf{0}$, computes column vectors $\mathbf{s}, \mathbf{t} \in \mathbb{R}^2$ which define the angle bisector $\mathcal{W} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{s} + \tau\mathbf{t},\ \tau \in \mathbb{R}\}$ between $\mathcal{G}_1$ and $\mathcal{G}_2$. Additionaly, plot $\mathcal{G}_1$, $\mathcal{G}_2$ and the angle bisector $\mathcal{W}$.

Hint: A template for the function `Problem2b` is available online and can be downloaded from course website.

Hint: Remember that you can use the results from Problem Sheet 2 for both of these problems.

3. Consider a line

$$\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{p} + \tau\mathbf{q},\ \tau \in \mathbb{R}\}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$, $\mathbf{q} \neq \mathbf{0}$ are given column vectors, and a circle

$$\mathcal{C} = \left\{\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : (x_1 - c_1)^2 + (x_2 - c_2)^2 = r^2\right\},$$

given by its centre $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and radius $r > 0$. Write a MATLAB function

```
function intersection = Problem3(r, centre, p, q)
```

that computes points of intersection of $\mathcal{G}$ and $\mathcal{C}$.
Use a stable implementation when solving the governing quadratic equation. Why is it impossible to properly handle the case when the intersection is (theoretically) only one point?

Hint: For the stable implementation use the function `zerosquadpolstab` (Code 1.8.10) from the lectures. A template for the function `Problem3` is available online and can be downloaded from course website.

4. (a) Consider two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ (column vectors), with $\mathbf{a} \neq \mathbf{b}$, and a line $\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{p} + \tau\mathbf{q},\ \tau \in \mathbb{R}\}$, given by column vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$, $\mathbf{q} \neq \mathbf{0}$. Consider a reflection line $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{s} + \tau\mathbf{t}, \tau \in \mathbb{R}^2\}$ across which $\mathbf{a}$ and $\mathbf{b}$ are reflected onto $\mathcal{G}$. When is there only one such $\mathcal{A}$ and when are there two such lines of reflection, depending on $\mathbf{a}, \mathbf{b}$ and $\mathcal{G}$?

(b) Create a MATLAB function

```
function [s, t] = Problem4(a, b, p, q)
```

Siehe nächstes Blatt!

that computes the line of reflection (by computing column vectors $\mathbf{s}$ and $\mathbf{t}$ which determine it) across which two given points, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ with $\mathbf{a} \neq \mathbf{b}$, are reflected onto a line $\mathcal{G}$, given through column vectors $\mathbf{p}$ and $\mathbf{q}$. Plot the points $\mathbf{a}, \mathbf{b}$, the line $\mathcal{G}$, the reflection line $\mathcal{A}$ and the reflection points. If $\mathbf{a}$ and $\mathbf{b}$ are on $\mathcal{G}$, the function should return $\mathbf{s} = \mathbf{p}$ and $\mathbf{t} = \mathbf{q}$.

Hint: This problem is related to Problem 2.(b). Notice that if the vector $\mathbf{b} - \mathbf{a}$ is not parallel to $\mathcal{G}$ then we have two lines of reflection (though you should only plot one of them). Otherwise, if the vector $\mathbf{b} - \mathbf{a}$ is parallel to $\mathcal{G}$ and if $\mathbf{a}, \mathbf{b} \notin \mathcal{G}$, then there is only one line of reflection. A template for the function `Problem4` is available online and can be downloaded from course website.

5. Create a MATLAB function

```
function [s, t]= Problem5(a1, a2, a3, p, q)
```

which for three distinct points (column vectors) $\mathbf{a}^1, \mathbf{a}^2$ and $\mathbf{a}^3 \in \mathbb{R}^2$ and a line $\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{p} + \tau\mathbf{q},\ \tau \in \mathbb{R}\}$, given by column vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$, $\mathbf{q} \neq \mathbf{0}$, computes the line of reflection $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{s} + \tau\mathbf{t}, \tau \in \mathbb{R}\}$, if such a line exists, that reflects $\mathbf{a}^1$ onto $\mathbf{a}^1$, and reflects the centroid $\mathbf{c} \in \mathbb{R}^2$ of the triangle $\triangle(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ onto $\mathcal{G}$. If such an $\mathcal{A}$ does not exist, the function should return $\mathbf{s} = \mathbf{0}$, $\mathbf{t} = \mathbf{0}$.

Hint: Draw the situation in this problem on a piece of paper. Notice that the solution exists only if $\|\mathbf{a}^1 - \mathbf{c}\| \leq \mathsf{dist}\,(\mathbf{a}, \mathcal{G})$, where $\mathbf{c}$ denotes the centroid of the triangle $\triangle(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$. How does this relate to Problem 3?
A template for the function `Problem5` is available online and can be downloaded from course website.

- **Abgabe der Serien:** Donnerstag, 10.09.2013 in der Übungsgruppe oder bis 16:00 Uhr in den Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.
- **Semesterpräsenz:** Aufgrund der grossen Nachfrage haben wir die Präsenz auf zwei Stunden ausgedehnt. Sie findet nun in einem besser geeigneten Raum statt am
Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13.
- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
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