Problem Sheet 5

Notation: Für eine quadratische Matrix $\mathbf{A}$ sind die Potenzen $\mathbf{A}^k$ definiert durch das k-fache Matrixprodukt:

$$\mathbf{A}^k := \underbrace{\mathbf{A} \cdot \dots \cdot \mathbf{A}}_{k \text{ Faktoren}} .$$

1. *(Online-Aufgabe)* Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, with $n \geq 3$. Determine which of the following statements are true and which ones are false. For the ones which are false try to find counterexamples.

1. If columns 1 and 3 of $\mathbf{B}$ are the same, so are columns 1 and 3 of $\mathbf{AB}$

(a) True.

(b) False.

2. If rows 1 and 3 of $\mathbf{B}$ are the same, so are rows 1 and 3 of $\mathbf{AB}$

(a) True.

(b) False.

3. If rows 1 and 3 of $\mathbf{A}$ are the same, so are rows 1 and 3 of $\mathbf{AB}$

(a) True.

(b) False.

4. $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$

(a) True.

(b) False.

Let $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$. Determine which of the following statements are true and which ones are false.

5. If $\mathbf{A}^2$ is defined then $\mathbf{A}$ is necessarily a square matrix.

(a) True.

(b) False.

6. If $\mathbf{AB}$ and $\mathbf{BA}$ are defined then $\mathbf{A}$ are $\mathbf{B}$ are both square matrices.

(a) True.

(b) False.

7. If $\mathbf{AB}$ and $\mathbf{BA}$ are defined then $\mathbf{AB}$ are $\mathbf{AB}$ are both square matrices.

(a) True.

(b) False.

8. If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the corresponding identity matrix.

(a) True.

(b) False.

2. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}.$$

a) Add $\mathbf{AB}$ to $\mathbf{AC}$ and compare with $\mathbf{A}(\mathbf{B}+\mathbf{C})$.

b) Multiply $\mathbf{A}$ times $\mathbf{BC}$. Then multiply $\mathbf{AB}$ times $\mathbf{C}$.

3. (i) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{asnd} \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

a) Compute $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$ and $\mathbf{A}^2 - \mathbf{B}^2$.

b) Why are the two results from **a)** different from each other?

c) Find two 2×2 matrices $\mathbf{C}$ and $\mathbf{D}$ such that

$$(\mathbf{C}+\mathbf{D})(\mathbf{C}-\mathbf{D}) = \mathbf{C}^2 - \mathbf{D}^2.$$

(ii) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}.$$

Show that $(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$ is different from $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$. Write down the correct rule for $(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 + \underline{\qquad} + \mathbf{B}^2$.

4. Consider the matrices (vectors regarded as $n \times 1$-matrices!)

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & 3 \\ -1 & -2 & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

Compute the following matrix products, if they are defined, and for the ones which are not defined, provide an argument why is that so

$$\mathbf{AB},\ \mathbf{BA},\ \mathbf{Ax},\ \mathbf{A}^2,\ \mathbf{B}^2,\ \mathbf{yx},\ \mathbf{y}^\top\mathbf{x},\ \mathbf{xy}^\top,\ \mathbf{B}^\top\mathbf{y},\ \mathbf{y}^\top\mathbf{B}.$$

5. Consider an $n \times n$ matrix $\mathbf{A}$ and an $n \times n$ identity matrix $\mathbf{I}_n$. Let us define a matrix $\mathbf{A}_1 := \mathbf{A} - \mathbf{I}_n$. $\mathbf{A}^k$ can now be computed as follows

$$\mathbf{A}^k = (\mathbf{I}_n + \mathbf{A}_1)^k = \mathbf{I}_n + \binom{k}{1}\mathbf{A}_1 + \binom{k}{2}\mathbf{A}_1^2 + \ldots + \binom{k}{k}\mathbf{A}_1^k. \tag{1}$$

Let us consider now a specific example, take

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Show that in this example, we have $\mathbf{A}_1^k = \mathbf{0}$ for $k \geq 3$.

Bitte wenden!

b) Compute $\mathbf{A}^{10}$ by using formula (1).

6. (i) In each of the following problems create a MATLAB function which for a given integer `n` creates the following *sparse matrices*.

a) An $n \times n$ sparse matrix, whose non-zero entries form a pattern shaped like the letter Z, with entries as follows

$$\begin{aligned} a_{1j} &= 1, \quad \text{for } j = 1, \dots, n \\ a_{j,n+1-j} &= j, \quad \text{for } j = 1, \dots, n \\ a_{n,j} &= n, \quad \text{for } j = 1, \dots, n. \end{aligned}$$

All other entries of the matrix should be 0.

```
function Z = ZShaped( n )
```

b) An $n \times n$ sparse matrix, whose non-zero entries form a pattern shaped like the letter X, with entries as follows

$$\begin{aligned} a_{jj} &= 2, \quad \text{for } j = 1, \dots, n \\ a_{j,n+1-j} &= 2, \quad \text{for } j = 1, \dots, n. \end{aligned}$$

All other entries of the matrix should be 0.

```
function X = XShaped( n )
```

c) An $n \times n$ sparse, three-band matrix, with entries as follows

$$\begin{aligned} a_{jj} &= 1, \quad \text{for } j = 1, \dots, n \\ a_{j+1,j} &= 2, \quad \text{for } j = 1, \dots, n-1 \\ a_{j+2,j} &= 3, \quad \text{for } j = 1, \dots, n-2 \end{aligned}$$

All other entries of the matrix should be 0.

```
function T = ThreeBand( n )
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix}$$

(a) Z shaped

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

(b) X shaped

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{pmatrix}$$

(c) Three-band

Abbildung 1: Example of matrices for Problem 6.(i).a)-c), with `n=5`

Hint: Use MATLAB's `sparse` command for creating the matrices.

(ii) Consider each of the three sparse matrices that were introduced in part (i). Which of those matrices will yield another sparse matrix, when multiplied with itself?

(iii) For each of the three matrix types from part (i) create a MATLAB function which implements a matrix-vector multiplication, $\mathbf{Ax}$, *without creating the matrix itself*. The input argument should be the column vector `x` and the output is the product $\mathbf{y} = \mathbf{Ax}$. Write the following MATLAB functions

Siehe nächstes Blatt!

```
function xz = MultiplyZShaped(x)
```

and

```
function xx = MultiplyXShaped(x)
```

and

```
function xt = MultiplyTridiagonal(x)
```

- **Abgabe der Serien:** Donnerstag, 24.10.2013 in der Übungsgruppe oder bis 16:00 Uhr in den Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.
- **Semesterpräsenz:** Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13.
- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
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