Problem Sheet 8

1. Through n measurements, where $n \in \mathbb{N}$, of the same quantity x we have obtained n approximate values $m_1, m_2, \ldots, m_n$. We want to retrieve the value of x by solving an overdetermined linear system by using the method of least squares.

a) Revise the section 3.9 from the lectures concerning the least squares solution of an over-determined system.

b) Write the linear system which governs this problem.

c) Set up the normal equations, according to the lectures (3.9.2).

d) Find the least squares solution of the problem, according to defintion 3.9.2.C, and give an interpretation of your results.

2. In this problem our aim is to use linear regression on a problem for which such an approach does not initially seem admissible. With a bit of foresight, one can see that using the appropriate inverse function can sometimes solve the issue at hand.

We consider a function $f(t) = \alpha e^{\beta \cdot t}$, where $\alpha > 0, \beta \in \mathbb{R}$ are unknown parameters. Given

$$f(t_i) = y_i > 0, \ i = 1, \ldots, n, \tag{1}$$

we will find estimates for α, β using an appropriate method of (linear) least squares.

a) Which non-linear system of equations can be derived from (1)?

b) Derive the overdetermined linear system by suitably linearising the problem.

c) Determine the normal equations which arise from the overdetermined system given in **b)**.

d) Create a MATLAB function

```
function [alpha,beta] = ExpoFuncFit(t,y)
```

which for column vectors `t` and `y` computes α and β by using the method of least squares.
Hint: You may use (and modify) the code 3.9.1 `linearregression.m` from the lectures

3. This problem is concerned with computing unknown parameters from a set of (specific) measured values.

A function $f(x)$ is a a linear combination of functions

$$g_1(x) = 2^x, \text{ and } g_2(x) = 2^{-x}.$$

The graph of $f(x)$, that is, the set

$$\Gamma(f) := \{(x, f(x)) : x \in \mathbb{R}\},$$

has been measured to go through points

Bitte wenden!

x_i	-2	-1	0	1	2
y_i	-8	-4	-2	4	12

in the (x, y) plane.

a) Write the overdetermined linear system for this problem.

b) Derive the normal equations for this system.

c) Use the method of least squares to find an approximation to $f(x)$.

4. For an $n \in \mathbb{N}$ we are given some points on a road $p_1 < p_2 < \ldots < p_n$, where $p_i \in \mathbb{R}$, $i = 1, \ldots, n$, the following sections of the road have been (possibly imprecisely) measured

$$p_i - p_j = d_{ij}, \quad \text{for } 1 \leq j < i \leq n.$$

Look at figure (1). The following problems are about formulating and solving a least squares

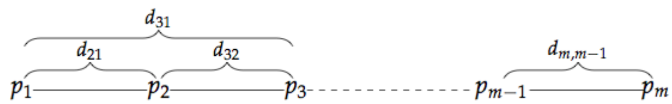

Abbildung 1

problem of determining $p_1, \ldots, p_n$ from those measurements.

a) How many equations, i.e. values of d_{ij} are we given?

b) State the previously described situation as an overdetermined linear system, with a suitable matrix $\mathbf{A}$ and right-hand side vector $\mathbf{b}$.

c) Determine $\ker(A)$, interpret the result and describe its consequences to the least-squares solution (look at theorem 3.9.2.E). Provide a modification of this problem that would ensure the uniqueness of its least-squares solution, and write down the appropriate overdetermined linear system.

d) Write the normal equations when $n = 5$ for the modified overdetermined system defined in **c)**.

e) Create a MATLAB function

```
function p = RoadLengths(D)
```

which given the measurements, contained in matrix D, computes the solution of the least squares problem.

5. In this problem we will show some equivalences

Consider a matrix $\mathbf{A} \in \mathbb{R}^{n,m}$ with $n > m$ and $\operatorname{rank}(\mathbf{A}) = m$ and a vector $\mathbf{b} \in \mathbb{R}^n$.

a) Show that $\mathbf{x} \in \mathbb{R}^m$ is a solution of $\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b}$ if and only if there is a vector $\mathbf{r} \in \mathbb{R}^n$ such that

$$\begin{pmatrix} \mathbf{A}^\top & \mathbf{0} \\ \mathbf{I} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \end{pmatrix}. \tag{2}$$

b) What is the relationship between the vector $\mathbf{r}$ and the least squares solution $\mathbf{x}$? You may find it beneficial to consult the section 3.9.3 of the lectures.

c) Are there any benefits in considerations of (2) instead of the usual normal equations

Hint: Consider problems **4.d)** and **4.e)**, with a large number of measurement points.

6. We have a discrete-time signal, given by a sequence of values $x_1, \ldots, x_m$, before and after which there is a "radio silence". These signals are transmitted through a channel with crosstalk, so that the signal received on the other end of the channel consits of a sequence of values $y_1, \ldots, y_m$. The relationship between the received and transmitted signals through the channel

$$0, 0, x_m, \ldots, x_1, 0, 0 \longrightarrow \boxed{\text{Channel}} \longrightarrow y_m, \ldots, y_1$$

is (approximately) given by the following series of equations.

$$y_i = \beta x_{i-1} + \alpha x_i + \beta x_{i+1},$$

where we take $x_0 = x_{m+1} = 0$. In the following we will talk about determining paramaters α and β from the measured signals by solving an overdetermined system of linear equations using a method of least squares.

a) Determine the overdetermined linear system for this problem.

b) Determine the corresponding normal equations.

c) Create a MATLAB function

```
function [beta, alpha] = CrosstalkChannel(x, y)
```

which for given $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ estimates α and β using the method of least squares.

- **Abgabe der Serien:** Donnerstag, 21.11.2013 in der Übungsgruppe oder bis 16:00 Uhr in den Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.
- **Semesterpräsenz:** Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13. Falls keine grosse Nachfrage besteht, warten die Assistenten maximal eine halbe Stunde. Wir bitten Sie deshalb, bei Fragen so früh als möglich zu erscheinen.
- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
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