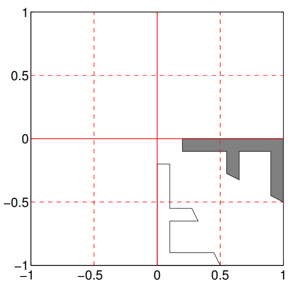
Problem Sheet 9

Problems **1.-6.** are about *linear maps* and their *matrix representations* with respect to some basis. The theory behind those has been covered in sections 4.1 and 4.2 of the lectures.

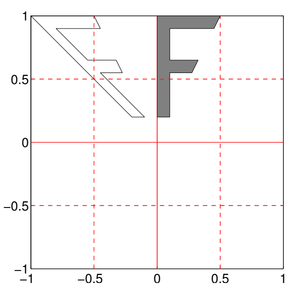
1. In section 4.1 of lecture notes we have seen the effect of some linear $\mathbb{R}^2 \to \mathbb{R}^2$ maps, visually demonstrated by their actions on certain figures. In this problem the goal is to read out the linear map which underlies the given transormation.

a) Indicate which of the given matrices is responsible for transforming the grey F area into the white F area.



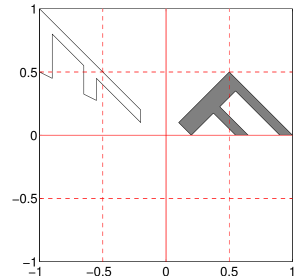
○ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b) Indicate which of the given matrices is responsible for transforming the grey F area into the white F area.



○ $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

c) Indicate which of the given matrices is responsible for transforming the grey F area into the white F area.



○ $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ ○ $\begin{pmatrix} -1 & -1 \\ 0.5 & 1.5 \end{pmatrix}$ ○ $\begin{pmatrix} -1 & -1 \\ 0.5 & -0.5 \end{pmatrix}$

2. In this task we go back to the matrix representation of a linear map and its associated subspaces, which were introduced in Theorem 4.1.C. in the lecture notes.

In the following subproblems we consider the linear map

$$\mathsf{F} : \mathbb{R}^3 \to \mathbb{R}^2,$$

which maps the Cartesian basis vectors (of unit length) $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \in \mathbb{R}^3$ onto vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3 \in \mathbb{R}^2$. Vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3 \in \mathbb{R}^2$ are given in figure 1. In summary, we have

$$\mathsf{F}\mathbf{e}^i = \mathbf{a}^i, \quad i \in \{1, 2, 3\}.$$

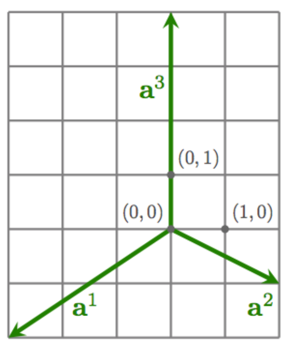


Abbildung 1: $\mathbf{a}^i = \mathsf{F}\mathbf{e}^i$, $i \in \{1, 2, 3\}$

(a) Which of the following matrices $\mathbf{A}$ represents the mapping F (with respect to the Cartesian bases of $\mathbb{R}^3$ and $\mathbb{R}^2$)?

○ $\mathbf{A} = \begin{pmatrix} -2 & -1 & 4 \\ -3 & 2 & 0 \end{pmatrix}$ ○ $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -2 & -3 \\ 4 & 0 \end{pmatrix}$

○ $\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}$ ○ $\mathbf{A} = \begin{pmatrix} -3 & 2 & 0 \\ -2 & -1 & 4 \end{pmatrix}$

(b) We have $\mathrm{im}(\mathsf{F}) = \mathbb{R}^2$.

○ True. ○ False.

(c) We have $\ker(\mathsf{F}) = \{\mathbf{0}\}$.

○ True. ○ False.

(d) F is an invertible mapping.

○ True. ○ False.

3. Once again we will examine the notion of a linear mapping given in Section 4.1. of the lecture and the corresponding matrix representation.

Take $\mathbf{x} \in \mathbb{R}^2$. Consider the linear map F on $\mathbb{R}^2 \to \mathbb{R}^2$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \overset{\mathsf{F}}{\mapsto} \quad \mathbf{x}' = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} .$$

(a) Give a geometric interpretation of this mapping, that is, describe its action on points in the plane.

(b) Show that F is a linear mapping.
Hint: You only need to show that F satisfies conditions (L1) and (L2) of Definition 4.1.A.

(c) Determine the matrix representation $\mathbf{A}$ of F with respect to the Cartesian basis.
Hint: Recall the equation in Definition 4.2.C., regarding the representation matrix of a linear map.

4. In Section 4.2 of the lectures we discussed the matrix representation of the differentiation map in the vector space of polynomials. This problem is concerned with a different linear map, and its matrix representation, given on the space of polynomials.

Let $\mathbb{P}_d$ be the vector space of polynomials of degree less or equal than d. The dimension of $\mathbb{P}_d$ is $n := d + 1$. The set of polynomials $\{1, x, x^2, \ldots, x^d\}$ constitutes a basis of $\mathbb{P}_d$. This is the so-called monomial basis, and we will adhere to it in the remainder of this problem.

Consider the mapping

$$\mathsf{F} : \mathbf{p}(x) \in \mathbb{P}_d \mapsto \mathbf{q}(x) = (x-1)\mathbf{p}'(x) \in \mathbb{P}_d(x) ,$$

which maps $\mathbf{p}(x)$ to the polynomial $\mathbf{q}(x) = (x-1)\mathbf{p}'(x)$, where $\mathbf{p}'(x)$ denotes the derivative of $\mathbf{p}(x)$.

(a) Revise the discussion of the matrix representation of the differentiation map on the space of polynomials, given in the lectures.

(b) Show that F is a linear map on $\mathbb{P}_d \to \mathbb{P}_d$.
Hint: You need to show that conditions (L1) and (L2) from Definition 4.1.A. hold.

(c) Determine the $n \times n$ matrix representation of F with respect to the monomial basis.
Hinweis: Recall that you should first determine the images of the basis vectors and their representation (i.e. coordinates) with respect to the basis of the codomain. These will consitute the columns of the corresponding matrix representation.

5. In this problem you are asked to determine the linear map hiding behind a recursively defined sequence, and find its matrix representation.

The famed Fibonacci sequence $(F_0, F_1, F_2, F_3, F_4, F_5, F_6, \dots) = (0, 1, 1, 2, 3, 5, 8, \dots)$ is typically defined as follows:

$$F_0 := 0,\ F_1 := 1\ , \quad F_{n+1} := F_n + F_{n-1}, \quad n \in \mathbb{N}.$$

Let us define $\mathbf{x}^n = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} \in \mathbb{R}^2$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

(a) We have a linear map H through which $\mathbf{x}^n$ and $\mathbf{x}^{n+1}$ are related through the equation

$$\mathbf{x}^{n+1} = \mathsf{H}\left(\mathbf{x}^n\right), \quad \text{for all } n \in \mathbb{N}_0.$$

Write down H.

(b) Determine the matrix representation $\mathbf{A}$ of $\mathsf{H} : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the Cartesian basis.

6. Condition (L1) of the Definition 4.1.A. states that every linear mapping maps ... to .... Considerations of that condition are sufficient to answer the following question:

Does there exist a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps an equilateral triangle whose centroid is at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to a right-angled triangle which has $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as one of its vertices?

In the following two problems we will discuss the *change of basis*, which was covered in sections 4.3 and 4.4 of the lectures.

7. In this problem we are concerned with the material covered in Section 4.4 of the lectures.

Consider a linear mapping $\mathsf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ which, with respect to the basis $\mathcal{B}$ of $\mathbb{R}^3$ where

$$\mathcal{B} := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

is represented by the matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The goal of this task is to determine $\widetilde{\mathbf{D}}$, which is the matrix representation of F with respect to the Cartesian basis

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

.

(a) Revise the section 4.4 of the lectures.

(b) Determine the change of basis matrix $\mathbf{S}$ on $\mathbb{R}^3$ (considered as the definition space of F) from the basis $\mathcal{B}$ (old basis) to the basis $\mathcal{E}$ (new basis).

(c) Determine the change of basis matrix $\mathbf{S}$ on $\mathbb{R}^3$ (considered as the image space of F) from the basis $\mathcal{B}$ (old basis) to the basis $\mathcal{E}$ (new basis).

(d) Write down $\widetilde{\mathbf{D}}$, the matrix representation of the mapping F with respect to the Cartesian basis $\mathcal{E}$, by using matrices $\mathbf{S}$ and $\mathbf{R}$ from the preceding subproblems.

(e) Determine the matrix $\widetilde{\mathbf{D}}$ with the help of **(c)**.

8. We consider the linear map $\mathsf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$, which has a matrix representation

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the Cartesian basis. Determine $\widetilde{\mathbf{A}}$, the matrix representation of F with respect to the basis

$$\widetilde{\mathcal{B}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

Hint: Think carefully before solving this problem, you can significantly shorten your calculations.

9. We will often have a linear mapping described not by a matrix but by other means. For $\mathbf{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3 \setminus \{0\}$ we define the maps

$$\mathsf{F} : \mathbb{R}^3 \to \mathbb{R}^3 \quad , \quad \mathsf{F}(\mathbf{x}) = \mathbf{a} \times \mathbf{x} \,,$$

where $\times$ denotes the vector product, as in Section 1.6 of the lectures.

a) Revise the properties of the vector product.

b) Show that F is a linear mapping.

c) Determine the matrix representation of F with respect to the Cartesian basis. What special property does this matrix representation have?

Hint: Recall that you need to determine the images of the basis vectors under F and their coordinates. These constitute the columns of the coresponding matrix representation.

d) Determine $\ker(\mathsf{F})$.

Hint: In order to determine the kernel of $\ker(\mathsf{F})$ you can either directly see it, or determine it by computing the kernel of the corresponding matrix representation.

e) Determine $\mathrm{rank}(\mathsf{F})$.

f) Determine $\langle \mathbf{x}, \mathsf{F}(\mathbf{x}) \rangle$, $\mathbf{x} \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle$ is the (Euclidean) inner product on $\mathbb{R}^3$.

10. Sometimes the mapping we encounter are described by a certain procedure, e.g., by a piece of code in some programming language. In these cases it is often not that easy to determine if the underlying mapping is linear, and the matrix representation associated with it.

In code 1 you will find six MATLAB functions, each one having a column vector `x` as an argument and a column vector `y` as the return. The MATLAB source codes in question can also be downloaded from the course website.

a) For each MATLAB functions determine the vector space in which `y` lies (depending on the size of `x`).

b) Which of these functions do **not** describe a linear mapping `x` $\mapsto$ `y`? Justify your answers.

Hint: You can use MATLAB to show which maps are **not** linear.

c) Determine their matrix representation with respect to the Cartesian basis of the functions which dovrepresent a linear map `x` $\mapsto$ `y` .

Hint: This matrix representation can be determined with the help of MATLAB , when `x` has a predetermined length, by taking into account that the j-th column of that matrix contains the coordinates of the image of the j-th unit vector, from the Cartesian basis, under that given matrix.

Listing 1: Linear or not?

```
function y = a9_f1(x)
  y = (1:length(x))'.*x(end:-1:1);
end

function y = a9_f2(x)
  y = x*x'*(1:length(x))';
end

function y = a9_f3(x)
  n = length(x); y = [];
  m = floor(n/2);
  for l=1:m, y = [y;x(l) - l*x(2*l)]; end
end

function y = a9_f4(x)
  n = length(x);
  y = (diag(x)+ones(n,1)*ones(1,n))*(n:-1:1)';
end

function y = a9_f5(x)
   n = length(x);
   a = 2.^(0:n-1); a = [a,a,a];
   for l=1:n
     y(l) = 0;
     for k=1:n, y(l) = y(l) + x(k)*a(n-l+k); end
   end
end

function y = a9_f6(x)
  n =length(x);
  y = (1:n)*sparse([(1:n-1)';(1:n)';(2:n)'],...
                   [(2:n)';(1:n)';(1:n-1)'],...
                   [x(1:n-1);x;x(2:n)],n,n);
end
```

- **Abgabe der Serien:** Donnerstag, 21.11.2013 in der Übungsgruppe oder bis 16:00 Uhr in den

Siehe nächstes Blatt!

Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.

- **Semesterpräsenz:** Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13. Falls keine grosse Nachfrage besteht, warten die Assistenten maximal eine halbe Stunde. Wir bitten Sie deshalb, bei Fragen so früh als möglich zu erscheinen.

- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
  www.math.ethz.ch/education/bachelor/lectures/hs2013/other/linalgnum_BAUG