Problem Sheet 10

1. This problem is about the details of the proof of Theorem 4.5.E from lectures.

Let V be a vector space such that $\dim(V) = n$ for $n \in \mathbb{N}$. Consider a projection $\mathsf{P} \in \mathcal{L}(V, V)$ for which the following holds

$$\begin{aligned} \left\{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^r\right\} \subset V &\quad \text{is a basis of } \operatorname{im}(\mathsf{P}), \\ \left\{\mathbf{b}^{r+1}, \mathbf{b}^{r+2}, \ldots, \mathbf{b}^n\right\} \subset V &\quad \text{is a basis of } \ker(\mathsf{P}), \end{aligned}$$

where $1 \leq r \leq n$.

a) Revise the concepts of the basis of a vector space (Definition 2.1.A) and linear independency (Definition 2.5.A).

b) Show that $\left\{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\right\}$ is a linearly independent set of vectors in V.

Hint: Write the appropriate linear combination of those vectors as given by the definition of linear independence and apply the projection operator P. Then use Theorem 4.5.B from the lectures and properties therein.

c) Show that $\left\{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\right\}$ is a basis of V.

2. Consider the following subspaces of $\mathbb{R}^3$:

$$\begin{aligned} U &:= \left\{ \mathbf{x} \in \mathbb{R}^3 \,:\, \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{x} \right\rangle = 0 \right\}, \\ W &:= \mathsf{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

a) Show that U is a subspace of $\mathbb{R}^3$ by verifying the requirements of Definition 1.3.C from the lectures.

b) Find a basis for U.

Hint: Bear in mind that a basis is not unique.

c) Find a basis of $\mathbb{R}^3$ with respect to which a projection onto U that is parallel to W and has the canonical matrix representation given by Theorem 4.5.E.

Hint: For the projection $\mathsf{P} : \mathbb{R}^3 \to \mathbb{R}^3$ on U parallel to W we have

$$\ker(\mathsf{P}) = W \quad \text{and} \quad \operatorname{im}(\mathsf{P}) = U\,.$$

d) Determine the matrix representation $\mathbf{P}$ of the projection P onto U that is parallel with W with respect to the Cartesian basis.

Hint: Since $\mathsf{P} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map, this subproblem can be seen as a change of basis of $\mathbb{R}^3$, as in Section 4.4 of the lectures.

3. We have introduced orthogonal matrices and isometries of a Euclidean vector space $\mathbb{R}^n$ in Section 4.6.2 of the lectures. Now we want to do some examples regarding those notions.

Given a matrix

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{3}} & a & b \\ \frac{1}{\sqrt{3}} & -c & d \\ e & 0 & -f \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}_0^+ = [0, \infty)$, solve the following problems.

a) How many (different) equations for unknowns a, b, c, d, e and f are implied by the orthogonality condition?

Hint: Note that for $\mathbf{Q} \in \mathbb{R}^{3,3}$ the matrix $\mathbf{Q}\mathbf{Q}^\top$ is symmetric.

b) Determine a, b, c, d, e and f such that $\mathbf{B}$ is an orthogonal matrix.

4. In Sections 4.6.3 and 4.6.4 of the lectures we considered fundamental isometries; reflections and rotations. In this problem, we will treat specific examples of reflections and rotations in $\mathbb{R}^3$, and their corresponding matrix representations.

We have three mappings in $\mathbb{R}^3$, $\mathsf{F}_1, \mathsf{F}_2$ and F_3, defined as follows

$$\begin{aligned} \mathsf{F}_1 &: \text{ reflection around the plane } \{x_1 = x_2\}. \\ \mathsf{F}_2 &: \ 45^\circ \text{ rotation around the axis } x_1. \\ \mathsf{F}_3 &: \ 30^\circ \text{ rotation around the axis } x_2. \end{aligned}$$

a) Find the matrix representations of (linear) mappings $\mathsf{F}_1, \mathsf{F}_2$ and F_3, with respect to the Cartesian basis of $\mathbb{R}^3$.

Hint: As in the Section 4.2. of the lectures, the columns of those matrix contain the coordinates of the images of basis vectors under each respective linear mapping.

b) Determine the matrix representation of the mappings $\mathsf{F}_2 \circ \mathsf{F}_1$ and $\mathsf{F}_3 \circ \mathsf{F}_2$.

Hint: Use Theorem 4.2.E from the lectures.

5. We want to identify when are linear mappings isometries by using their matrix representations.. You might find it beneficial to do Problem 3 before doing this problem.

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & s & t \\ 0 & 0 & -1 \\ r & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Determine $s, t, r \in \mathbb{R}$ so that the correponding linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ is an isometry.

Hint: Use Theorem 4.6.2.B from the lectures.

6. This problem is about a special class of isometries of the plane and their matrix representations.

Consider the following 2×2 matrices

$$\mathbf{A} = \mathbf{I}_2 - 2\mathbf{u}\mathbf{u}^\top, \mathbf{B} = \mathbf{I}_2 - 2\mathbf{v}\mathbf{v}^\top,$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ satisfy $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. These matrices define two linear mappings, F and G.

Siehe nächstes Blatt!

a) Show that F and G are isometries.

b) For both F and G there is a one-dimensional subspace of $\mathbb{R}^2$ whose vectors are mapped by F, respectively G, to themselves. Describe those subspace with a formula.

c) Provide a geometrical interpretation of maps F and G and illustrate your findings with a sketch.

d) Show that applying F and G one after the other is actually equivalent to a rotation. What is the angle of this rotation?

Hint: Compute the matrix representation of the linear map $\mathsf{G} \circ \mathsf{F}$ and then refer to Definition 4.6.4.1 from the lectures.

e) Give a geometrical interpretation of the preceding subproblem.

7. An object casting a shadow in front a source of light is a real-life example of a projection. We will explore this phenomenon in this problem, with further assistance from MATLAB.

Let us take the object casting the shadow to be the wire tetrahedron, i.e., the set of edges of a tetrahedron. A tetrahedron is a triangular pyramid with 4 vertices, $\binom{4}{2} = 6$ edges and $\binom{4}{3} = 4$ (triangular) faces. In a three-dimensional space, a tetrahedron is uniquely described by 4 vertices that do not all lie in the same plane.

Consider now a tetrahedron given by four vertices $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4 \in \mathbb{R}^3$ (coordinates given with respect to the Cartesian basis of $\mathbb{R}^3$). Assume that the last (third) coordinate of all 4 points is a positive number. The incident beam of light has a direction parallel to

$$\mathbf{u} = \begin{pmatrix} \cos\phi\cos\vartheta \\ \sin\phi\cos\vartheta \\ \sin\vartheta \end{pmatrix}, \quad \phi \in [0, 2\pi[\,, \quad \vartheta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\,. \tag{1}$$

a) Show that the vector $\mathbf{u}$ from (1) is a unit vector in $\mathbb{R}^3$.

b) The class of maps $\mathsf{S}_{\phi,\vartheta} : \mathbb{R}^3 \to \mathbb{R}^2$, $\phi \in [0, 2\pi[$, $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\vartheta \neq 0$, is described as follows:

> $\mathsf{S}_{\phi,\vartheta}(\mathbf{z})$ is the first two coordinates of the projection of a point $\mathbf{z} \in \mathbb{R}^3$, $z_3 > 0$ onto the plane $\{x_3 = 0\} \subset \mathbb{R}^3$ that is cast by a light beam with a direction $\mathbf{u}$, which is defined by (1). The light beams are assumed to be parallel.

Determine the matrix representation of the map $\mathsf{S}_{\phi,\vartheta}$ with respect to Cartesian bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

c) Recall the functionality of MATLAB's `plot` function.

d) Why is `plot` appropriate for visualisation of the image of a tetrahedron by a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ (not taking into account the errors with respect to precision).

Hint: Theorem 4.1.F. from the lectures.

e) Create a MATLAB function

```
function plottetshadow(A,phi,theta)
```

that computes and plots the shadow cast by a tetrahedron, determined by its vertices given as columns of a 3×4 matrix `A`, onto the plane $\{x_3 = 0\}$, by light beams that are parallel to $\mathbf{u}$, given by `phi` and `theta` through the equation (1).

Hint: For

$$\mathbf{a}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{a}^2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{a}^3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} , \quad \mathbf{a}^4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and $\phi = \frac{\pi}{3}$, $\vartheta = \frac{2\pi}{5}$ the output should resemble figure 1.

8. When is the shadow cast by our wire tetrahedron a triangle? Experiment with your implementation of `plottetshadow`.

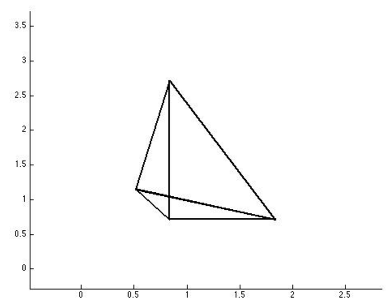

Abbildung 1: Shadow cast by a tetrahedron

- **Abgabe der Serien:** Donnerstag, 5.12.2013 in der Übungsgruppe oder bis 16:00 Uhr in den Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.
- **Semesterpräsenz:** Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13. Falls keine grosse Nachfrage besteht, warten die Assistenten maximal eine halbe Stunde. Wir bitten Sie deshalb, bei Fragen so früh als möglich zu erscheinen.
- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
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