Intermezzo on numerical linear algebra

$\underline{\underline{A}}$ $n \times n$ matrix or $m \times n$ $\quad \underline{\underline{A}} = [A_{:,1}, A_{:,2}, \dots, A_{:,n}]$

↑ column vectors

$$\underline{\underline{A}}\,\underline{x} = \sum_{i=1}^{n} x_i A_{:,i}$$

orthogonal vectors: $\underline{q}_1, \underline{q}_2, \dots, \underline{q}_n$ if

$$\underline{q}_i^T \underline{q}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

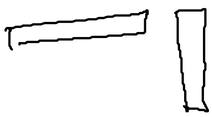
$$\underline{\underline{Q}} = [\underline{q}_1 \ \underline{q}_2 \ \cdots \ \underline{q}_n] \text{ orthogonal matrix}$$

$\underline{\underline{Q}}^T \underline{\underline{Q}} = I_n$, hence inverse of $\underline{\underline{Q}}$ is just $\underline{\underline{Q}}^T$

Important: ortogonal transformations do not change lengths!

$$\|\underline{\underline{Q}}\,\underline{x}\|_2^2 = (\underline{\underline{Q}}\,\underline{x})^T (\underline{\underline{Q}}\,\underline{x}) = \underline{x}^T \underbrace{\underline{\underline{Q}}^T \underline{\underline{Q}}}_{I} \underline{x} = \underline{x}^T \underline{x} = \|\underline{x}\|_2^2$$

$$\|v\|_2^2 = v_1^2 + \dots + v_n^2 = \underline{v}^T \underline{v}$$



Important decompositions of a matrix $\underline{\underline{A}}$:

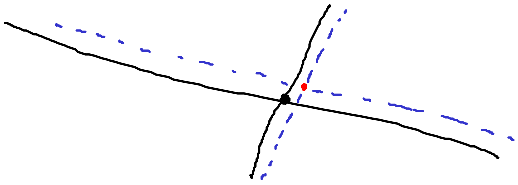
(1) Gauss-elimination is written and used as the LU-decomp.:

$$\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ * & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$$

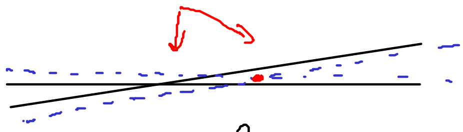
condition number of a matrix

$$\text{cond}_2(\underline{\underline{A}}) = \|\underline{\underline{A}}\|_2 \, \|\underline{\underline{A}}^{-1}\|_2$$ computed in a different way!

$$\|\underline{\underline{A}}\|_2 := \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$



$cond_2(A)$ is "small"



$cond_2(A)$ large
$Ax=b$ is "bad conditioned"

② QR-decomposition

$\underline{\underline{A}} = \underline{\underline{Q}}\ \underline{\underline{R}}$ → upper triangular matrix

$\underline{\underline{Q}}$ ↓ orthogonal matrix

$\underline{\underline{A}}x=b \iff QRx=b \;\Big|\; Q^T \quad \Rightarrow \quad Rx = Q^Tb$

no errors here!

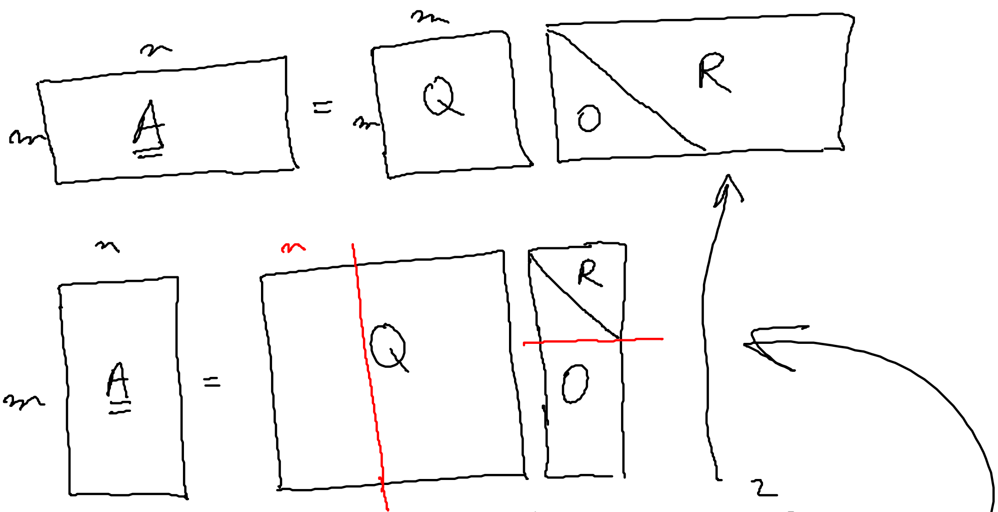
because Q is orthogonal, hence keeps lengths constant

$cond_2(Q) = 1$

$m \times n$ Matrix $\underline{\underline{A}}$

$m < n$

$m > n$



operations is $m^2 n$

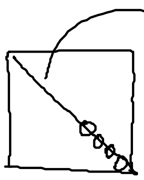
qr(A)
qr(A, econ = True) # operations mn^2

③ singular value decomposition

$\underline{\underline{A}} = U \Sigma V^T$ → orthogonal rows

U ↙ orthogonal columns

Σ ↘ diagonal matrix



$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$

$r = rank(A)$

$\sigma_1, \sigma_2, \ldots, \sigma_r$ = singular values of $\underline{A}$

$\underline{A} \in \mathbb{C}^{m \times n} \Rightarrow U \in \mathbb{C}^{m \times m}, \; V \in \mathbb{C}^{n \times n}$ unitary. (orth.)

$p := \min\{m, n\}$

$A = U \Sigma V^H, \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$

<u>Application</u>: Principal Component Analysis (Hauptkomponentenanalyse)

n Points $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n \in \mathbb{R}^m$

↪ Measurements

linear dependence?

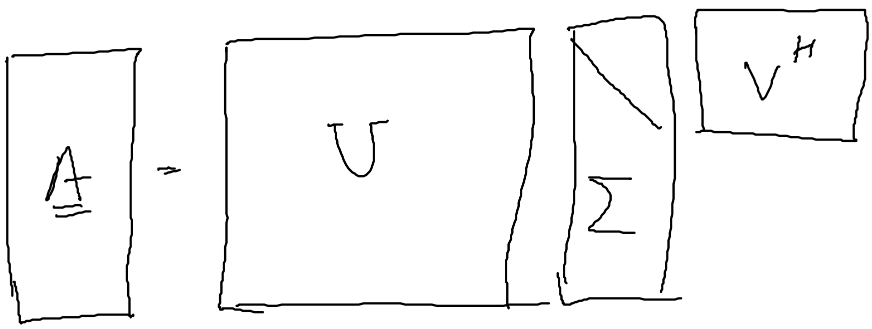
$\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ stay in a lower dimensional space of dimension $r \leq \min(m, n)$

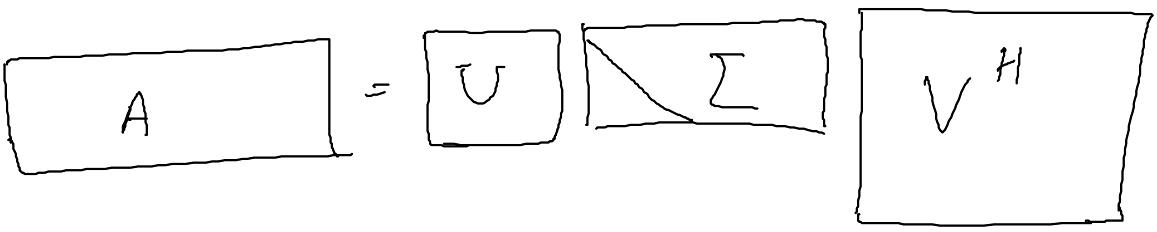
find r and an orthonormal basis of $\text{span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$

Linear algebra: $\underline{A} = [\underline{a}_1 \; \underline{a}_2 \ldots \underline{a}_n]$

$\text{rank}(\underline{A})$

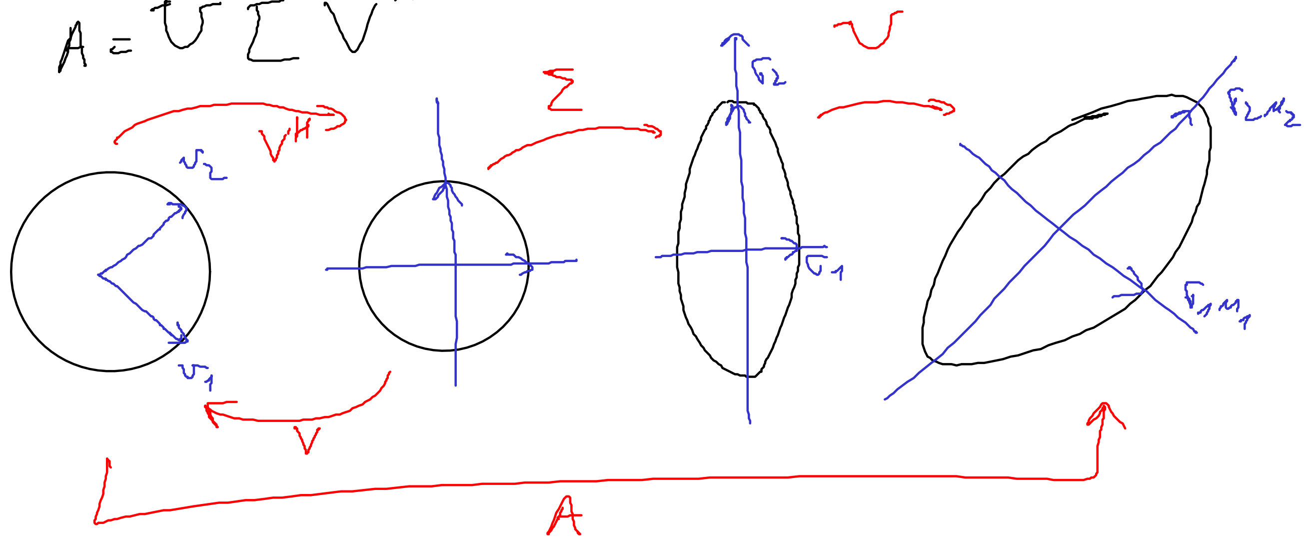
SVD gives us $r = \text{rank}(\underline{A})$ and basis!





Note SVD is not unique but singular values $\sigma_1, \dots, \sigma_p$ are unique!

$$A = U \Sigma V^H$$



$$\begin{cases} \underline{\underline{A}}\, \underline{v}_j = \sigma_j u_j & \text{for } j = 1, 2, \dots, r = \operatorname{rang}(A) \\ \underline{\underline{A}}\, \underline{v}_j = 0 & \text{for } j > r \end{cases}$$

$\underline{v}_j$ build a orthonormal basis in $\operatorname{Ker}(\underline{\underline{A}})$

$$\begin{cases} A^H u_j = \sigma_j v_j & \text{for } j = 1, 2, \dots, r \\ A^H u_j = 0 & \text{for } j > r \end{cases}$$

u_j build an orthonormal basis in $\operatorname{Im}(\underline{\underline{A}})$

Used for compression: (singular values)

$$\underline{\underline{A}} = U \Sigma V^H = \sum_{j=1}^{p} \sigma_j U_j \cdot V_j^H$$

Remark $\sigma_1^2 \geq \dots \geq \sigma_p^2 \geq 0$ are the eigenvalues of (EW)

$$A^H A, \quad A A^H, \quad \begin{bmatrix} 0 & A^H \\ A & 0 \end{bmatrix}$$

corresponding to eigenvectors (EV)

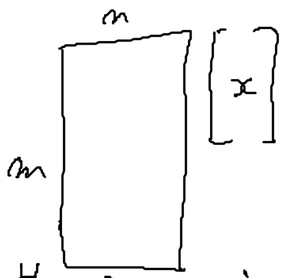
$v_{:,1}, \; v_{:,2}, \dots, \; v_{:,p}$

$u_{:,1}, \; u_{:,2}, \dots, \; u_{:,p}$

$$\begin{bmatrix} u_{:,1} \\ v_{:,1} \end{bmatrix}, \dots, \begin{bmatrix} u_{:,p} \\ v_{:,p} \end{bmatrix}$$

svd , an economical version of svd.

Minimization on a spere:



$\underline{A} \in \mathbb{R}^{m \times n}$, $m > n$

U orthogonal

$$\min_{\|x\|_2 = 1} \|\underline{A}\underline{x}\|_2^2 = \min_{\|x\|_2=1} \|U \Sigma \underbrace{V^H x}_{y}\|_2^2 = \min_{\|y\|_2 = 1} \|U \Sigma y\|_2^2 =$$

SVD of A: $U \Sigma V^H$

$\|y\|_2 = \|V^H x\|_2 = \|x\|_2 = 1$

V^H orthogonal matrix/transform

$$= \min_{\|y\|_2=1} \|\Sigma y\|_2^2 = \min_{\|y\|_2=1} (\sigma_1^2 y_1^2 + \dots + \sigma_n^2 y_n^2) = \sigma_n^2$$

decreasing.

attained for $\underline{y} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \underline{V}^H \underline{x} \Rightarrow \underline{x} = V \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = V_{:,n}$

(n^{th} colomn of V)

<u>Consequence</u> 1) $\|A\|_2 = \sigma_1(A)$

2) $\underline{A} \in \mathbb{R}^{n \times n}$ invertible $\Rightarrow$ $\text{cond}_2(A) = \frac{\sigma_1}{\sigma_n}$

↳ Nice way to compute the cond. number of a matrix!