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Algebraic Topology I
HS 2014

Solution 3

Fundamental groups-covering spaces

1 Use Van Kampen theorem to show that the fundamental group $\pi_1(X, x_0)$ of a CW-complex X with a single zero cell x_0 admits a presentation $< S|R >$ where S is the set of one cells of X and R is parametrized by the set of two cells of X.

Solution This is proven in [Hatcher, Proposition 1.26]

2 Let K be the Klein bottle and X the subspace of $\mathbb{R}^3$ consisting of a Klein bottle that self intersects in a circle (exercise 4, problem set 1).

(a) Determine the fundamental group of K and of X.

Solution Applying exercise 1 we get $\pi_1(K, x_0) =< a, b|aba^{-1}b >$. We know from exercise 4 problemset 1 that X is homotopy equivalent to $S^2 \vee S^1 \vee S^1$, in particular Van Kampen's theorem tells us that $\pi_1(X, x) = \mathbb{F}_2$.

(b) What is the homomorphism $f_* : \pi_1(K, p) \to \pi_1(X, f(p))$ induced by the natural projection $f : K \to X$?

Solution In order to prescribe the homomorphism it is enough to determine the image of the generators of the group. A representative β for the class b can be chosen to be the horizontal line in the square. Since $f(b)$ bounds a disc, we get that its image is trivial in the fundamental group. The curve $f(a)$, instead, is one of the two generators of the fundamental group $\pi_1(X, f(p))$.

(c) Is it injective? Is it surjective?

Solution The map is not injective nor surjective: the element b belongs to the kernel of f_*, and one of the two generators of $\mathbb{F}_2$ do not belong to the image.

3 Let X, Y be connected n-dimensional topological manifolds.

(a) If we denote by T^2 the torus $S^1 \times S^1$ determine what is $T^2 \# T^2$.

Solution $T^2 \# T^2$ is the surface of genus 2, Σ_2.

(b) If $n \geq 3$, and X, Y are connected, determine $\pi_1(X \# Y, x_0)$ knowing $\pi_1(X, x)$ and $\pi_1(Y, y)$.

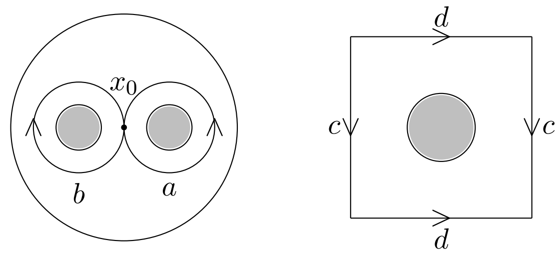
Solution It is an easy application of Van Kampen Theorem that if x_0 belongs to the boundary sphere than $\pi_1(X \# Y, x_0) = \pi_1(X, x_0) * \pi_1(Y, x_0)$. Indeed we can consider the two open sets U_1 that consists of the union of $\overline{X}$ and a small neighborhood V_Y of B_Y in $\overline{Y}$ with the property that $V_Y \setminus B_Y$ retracts on ∂B_Y, and U_2 consisting of the union of $\overline{Y}$ and a small open neighborhood V_X of B_X in $\overline{X}$ with the analogue property. The intersection $U_1 \cap U_2$ is connected and simply connected (since it retracts on S^{n-1} that is simply connected by our hypothesis on n), we get that the fundamental group of the space is the free product of the fundamental groups of the two open subsets $\overline{Y}$ and $\overline{X}$. Moreover since Y is obtained by $\overline{Y}$ by adding a (contractible) $n-$dimensional ball (resp. X from $\overline{X}$), one gets, applying Van Kampen once more that $\pi_1(X, x_0) = \pi_1(\overline{X}, x_0)$ and similarly for Y and $\overline{Y}$.

4 Let us denote by $\Sigma_{0,3}$ the complement in the sphere S^2 of three open discs, and by $\Sigma_{1,1}$ the complement in the torus $T = S^1 \times S^1$ of an open disc.

(a) Determine the fundamental groups $\pi_1(\Sigma_{0,3}, x)$ and $\pi_1(\Sigma_{1,1}, x)$.

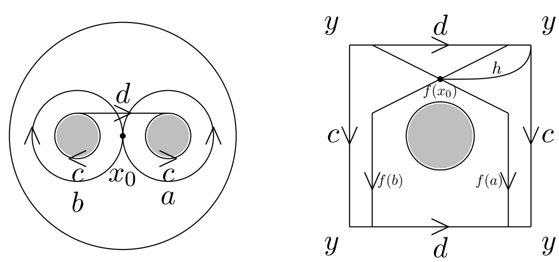
Solution. Both groups are free on two generators, as can be seen from the fact that the two surfaces retract on graphs. In fact the surface $\Sigma_{0,3}$ is homeomorphic to the complement in the disc D^2 of radius 5 in $\mathbb{R}^2$ of the discs with centers $(\pm 2, 0)$ and radius 1. It is easy to check that the surface $\Sigma_{0,3}$ deformation retracts on the wedge of the two circles with center $(\pm 2, 0)$ and radius 2. In particular the fundamental group of $\Sigma_{0,3}$ is free and generated by the loops a, b in the picture.

The torus with one disc removed, deformation retracts on its one skeleton, hence its fundamental group is free on two generators.



(b) Consider the map $f : \Sigma_{0,3} \to \Sigma_{1,1}$ obtained by identifying two boundary circles of $\Sigma_{0,3}$. Determine the map $f_* : \pi_1(\Sigma_{0,3}, x_0) \to \pi_1(\Sigma_{1,1}, f(x_0))$.

Solution. Denoting by a, b the generators of $\pi_1(\Sigma_{0,3}, x_0)$ and by c, d the generators of $\pi_1(\Sigma_{1,1}, y)$ in the picture, we claim that if h is the path pictured in the picture below, we have that $f_*a = \beta_h(c)$ and $f_*b = \beta_h(d^{-1}cd)$. In fact we can chose the map f so that the image of the two circles bounding the gray area in $\Sigma_{0,3}$ are mapped to the curve c. Under such a map the images of the curves a, b are shown in the second picture, and it is easy to check that $\beta_{\overline{h}} f_* a = c$ and $\beta_{\overline{h}} f_* b = d^{-1}cd$, by retracting the curves on the boundary.



(c) Is f_* injective? Is it surjective? You can use that any subgroup of a free group is free.

Solution. f_* is injective. In fact the subgroup of $\pi_1(\Sigma_{1,1}, x)$ generated by c and $d^{-1}cd$ is free on two generators: in fact it is free being a subgroup of a free group, it has at most two generators being the image of a group generated by two elements, and it has precisely two generators since the element $d^{-1}cd$ of $\mathbb{F}_2$ cannot be written as c^k for any k. The homomorphism f_* is not surjective. Assume by contraddiction that f_* is surjective, then also the induced homomorphism $\overline{f}_*: \mathbb{Z}^2 \to \mathbb{Z}^2$ would be surjective (where $\mathbb{Z}^2$ is the abelianization of $\mathbb{F}_2$). But that is not, since d is not in the image of $\overline{f}_*$.

5. Let $p : \overline{X} \to X$ be a covering space, and let A be a subspace of X. Denote by $\overline{A}$ the subset of $\overline{X}$ given by $\overline{A} = p^{-1}(A)$.

   (a) Show that the restriction $p : \overline{A} \to A$ is a covering space.

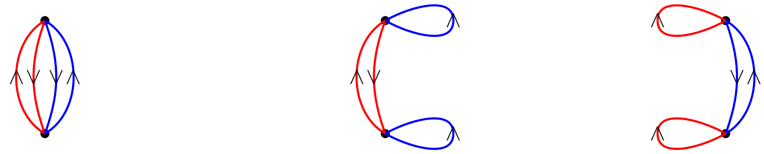
   Solution. Both spaces $\overline{A}$ and A are understood with the subspace topology. The restriction of p is clearly continuous. Moreover, since $p : \overline{X} \to X$ is a covering space, for each point $a \in A$ there exists an open neighborhood U of a such that $p^{-1}U$ is a disjoint union of open sets each of which is mapped by p homeomorphically to U. The open neighborhood $U \cap A$ of a in A clearly satisfies the same property. Therefore the restriction is a covering space.

   (b) Assume that $\overline{X}$ is the universal covering of X. Is $\overline{A}$ the universal covering of A?

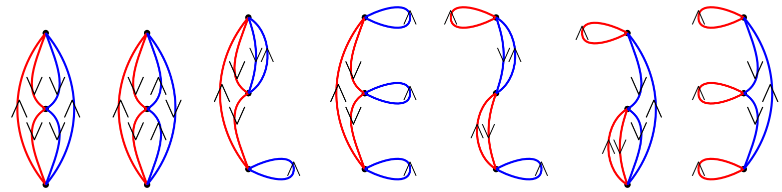
   Solution No, it is not: to get a counterexample we can take X to be equal to $\mathbb{R}^2$ and A to be the unit circle. It is not hard to show that under these hypotheses $\overline{A}$ is the universal cover of A if and only if the inclusion $i : A \to X$ induces an isomorphism on fundamental groups, that the set $\overline{A}$ is connected if and only if i_* is surjective, and that each component of $\overline{A}$ is simply connected if and only if i_* is injective.

6. Find all connected two sheeted and three sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces without basepoints.

   Solution Let us consider the CW structure on $S^1 \vee S^1$ consisting of a single zero cell, and two 1 cells with the only possible gluing map. A topological space that is the domain of a two sheeted covering map of X is a CW complex Y with two 0 cells and four 1 cells. In order to describe also the covering map it is useful to add arrows and colors to the 1 cells, this will determine for us uniquely what is the image of the cell under the covering map. It is easy to check that there are 4 coloured CW complexes with two 0 cells and four 1 cells, and only three such complexes cover X and are connected:

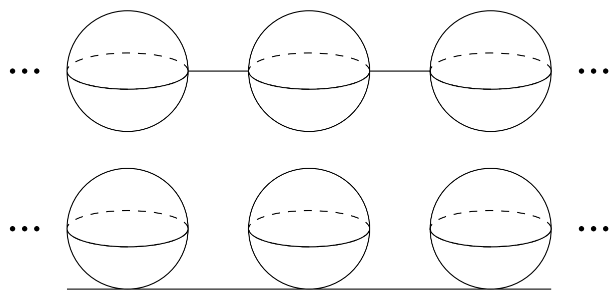


There are seven nonequivalent three sheeted coverings of $S^1 \vee S^1$, and they are shown in the picture:



In order to show that these are all, notice that the three sheeted coverings $p : \overline{X} \to X$ without basepoint are parametrized by the orbits of the action of the permutation group $\mathcal{S}_3$ on $\mathcal{S}_3 \times \mathcal{S}_3$ by conjugation on both factors. Indeed in order to prescribe a three sheeted covering it is enough to determine how do the two generators of the fundamental group of $S^1 \vee S^1$ act on the fiber (that is a set of cardinality 3). An isomorphism of covering spaces corresponds to a permutation of the fiber, and this correspond to the action of $\mathcal{S}_3$ on $\mathcal{S}_3 \times \mathcal{S}_3$ by conjugation on both factors. Up to conjugation we only care about the lengths of the cycles of the permutation corresponding to the first generator, that can either be a three cycle, or a transposition, or the trivial permutation. The first four coverings have a three cycle, the fifth and the sixth have a permutation, the last one has the trivial permutation. In order to determine the equivalence classes of coverings for which the image of the first generator is a three cycle, we should consider the equivalence classes for the action of the stabilizer of a three cycle (that is the group generated by the three cycle itself) on $\mathcal{S}_3$. There are four equivalent classes: the two three cycles are nonequivalent, all the permutations are equivalent, the trivial permutation gives the last equivalence class. If the image of the first generator is a transposition there are still four equivalence classes (two inequivalent transpositions, the three cycles, the trivial permutation) but only two give connected coverings. Only one of the three equivalence classes for the action of $\mathcal{S}_3$ that is the stabilizer of the identity give connected coverings: the class of the three cycles.

7. Consider the spaces X, Y of exercise 3, problem set 1, and the space $Z = S^1 \vee S^2$. Draw a picture of the universal coverings of X, Y and Z.
Solution Y is simply connected, in particular it is its own universal cover. For the spaces X, Z we have, for example:



In particular the universal covers of X and Y are homotopy equivalent, but not homeomorphic.

8. Let X, Y be connected, locally connected, semilocally simply connected topological spaces, and let $\widetilde{X}, \widetilde{Y}$ be the universal coverings.

   (a) Show that for every continuous map $f : X \to Y$ there exists a map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ making the following diagram commutative.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

   Solution We consider the map $g = f \circ p$ first:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow & \searrow^{g} & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

   Since $\widetilde{X}$ is simply connected the map $g_* : \pi_1(\widetilde{X}, x) \to \pi_1(Y, g(x))$ is injective, in particular g lifts to a map $\widetilde{f}$.

(b) Is such a map $\widetilde{f}$ unique?

Solution No, as soon as $\pi_1(Y)$ is not trivial, the map $\widetilde{f}$ is not unique. In fact the image under $\widetilde{f}$ of the basepoint x can be any element of the fiber $p^{-1}(g(x))$ and this consists of more than one point if $\pi_1(Y, y)$ is non trivial.

(c) Assume that f is injective, show that $\widetilde{f}$ is injective if and only if $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is injective.

Solution. Let us assume by contraddiction that $\widetilde{f}$ is not injective. Then there are two points x, y in $\widetilde{X}$ that have the same image. From the commutativity of the diagram and the injectivity of f we get that x, y belong to the same fiber of p. Let $\gamma : [0, 1] \to \widetilde{X}$ be a map with $\gamma(0) = x, \gamma(1) = y$. The loop $p \circ \gamma$ defines an element in the fundamental group $\pi_1(X, p(x))$. We claim that $[\gamma]$ belongs to the kernel of f_*: in fact $\widetilde{f} \circ \gamma$ is a closed loop in $\widetilde{Y}$ that is hence homotopically trivial (since $\widetilde{Y}$ is simply connected). In particular $f_*[p \circ \gamma] = [p \circ \widetilde{f} \circ \gamma] = p_*[\widetilde{f} \circ \gamma] = 0$ as an element of $\pi_1(Y, f(p(x)))$.

Viceversa assume f_* is not injective and let γ be a loop with $[\gamma] \in \ker f_*$. Let $\overline{\gamma}$ be a lift of γ to $\widetilde{X}$, the two (distinct) endpoints x, y of $\overline{\gamma}$ are mapped under $\widetilde{f}$ to the same point.