Exercise sheet 1

The content of the marked exercise **(*)** should be known for the exam.

1. Let $(G, \cdot)$ be a group. We say that G is *abelian* if $\forall x, y \in G$, $x \cdot y = y \cdot x$. For $g \in G$ we define the *order* of g, which we denote $\mathrm{ord}_G(g)$, as the minimal positive integer n such that $g^n = 1_G$, if such n exists. Else we say that g has infinite order. Prove the following statements for a group G:

1. If $e \in G$ is s.t. $\forall x \in G$, $e \cdot x = x$, then $e = 1_G$.
2. G is abelian if and only if the inversion map $G \to G$, $x \mapsto x^{-1}$ is a group homomorphism.
3. If $g^2 = 1_G$ for every $g \in G$, then G is abelian.
4. If $g \in G$ has finite order, g^{-1} is a power of g.
5. If G is finite, every $g \in G$ has finite order.

2. We will here consider *monoids*, which are defined in the same way as groups, but without inversion map. More precisely, a *monoid* consists of a set S together with a map $- \cdot - : G \times G \to G$ and a distinguished element $1_S \in S$ satisfying the following axioms:

- $\forall x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $\forall x \in S$, $1_S \cdot x = x \cdot 1_S = x$

We say that $y \in S$ is a *left* (resp., *right*) *inverse* of $x \in S$ if $y \cdot x = 1_S$ (resp., $x \cdot y = 1_S$).

Let X be a non-empty set and consider the set of functions $\mathrm{End}(X) = \{f : X \to X\}$.

1. Prove that $\mathrm{End}(X)$, together with the composition of functions $\circ$, is a monoid for every set X.
2. Prove that $f \in \mathrm{End}(X)$ has a left (resp., right) inverse if and only if f is injective (resp., surjective).
3. For which sets X does there exist $f \in \mathrm{End}(X)$ which has a left inverse but no right inverse?

[You can use this formulation of the axiom of choice: Let $\{X_i\}_{i \in I}$ be a family of non-empty sets indexed by $I \neq \varnothing$. Then there exists a family $\{x_i\}_{i \in I}$ such that $x_i \in X_i$]

Please turn over!

3. Show that there are precisely two non-isomorphic groups of order 4, and construct their multiplication table.

4. Consider the set $\mathbb{Z} \times \mathbb{Z}$ together with the binary operation $*$ defined by

$$(a, h) * (b, k) = (a + (-1)^h b, h + k)$$

1. Show that $(\mathbb{Z} \times \mathbb{Z}, *)$ is a group and that it is not abelian.
2. Find all elements having finite order.
3. Consider the projection maps $\pi_1, \pi_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\pi_1((m, n)) = m$ and $\pi_2((m, n)) = n$. Determine if they are morphism of groups $(\mathbb{Z} \times \mathbb{Z}, *) \to (\mathbb{Z}, +)$.

5. **(*)** Fix an integer $n > 1$ and consider the symmetric group $S_n := \mathrm{Sym}(\{1, \ldots, n\})$. For $p(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$ and $\sigma \in S_n$, define $p_\sigma = p(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$. Let $f := \prod_{1 \leq i < j \leq n}(X_i - X_j) \in \mathbb{C}[X_1, \ldots, X_n]$.

1. Prove that for every permutation $\sigma \in S_n$, there exists a unique element $\alpha(\sigma) \in \{\pm 1\}$ such that $f_\sigma(X) = \alpha(\sigma) f$.
2. Show that the resulting map

$$\alpha : S_n \to \{\pm 1\}$$

is a group homomorphism.

3. Let $a \neq b$ be elements of $\{1, \ldots, n\}$, and consider the permutation $\tau \in S_n$ switching a and b and fixing all the other elements. Show that $\alpha(\tau) = -1$.

Due to: 25 September 2014, 3 pm.