Exercise sheet 10

The content of the marked exercises **(*)** should be known for the exam.

1. **(*)** (Characterization of gcd and lcm in terms of principle ideals). Let A be a PID and take two non-zero elements $a, b \in A$. Show:

1. $aA + bA = dA$, where d is a *greatest common divisor* of (a, b) in the sense that
   a) $d|a$ and $d|b$, and
   b) for all $d' \in A$ s.t. $d'|a$ and $d'|b$, we have $d'|d$.
2. $aA \cap bA = mA$, where m is a *least common multiple* of (a, b) in the sense that
   a) $a|m$ and $b|m$, and
   b) for all $m' \in A$ s.t. $a|m'$ and $b|m'$, we have $m|m'$.
3. In the factorial ring $A = \mathbb{C}[X, Y]$ there are elements a and b which are irreducible, with $aA \neq bA$, but for which $aA + bA \neq A$.

2. Let A be a factorial ring.

1. Suppose that $a \in A \setminus A^\times$, $a \neq 0$, with $a = \prod_{i=1}^k r_i^{n_i}$ for some $k, n_i \in \mathbb{Z}_{>0}$ and some irreducible elements $r_i \in A$ such that $r_iA \neq r_jA$ for $i \neq j$. Prove that for every $b \in A$, we have that b divides a if and only if we can write
$$b = u \prod_{i=1}^k r_i^{m_i}, \text{ for some } u \in A^\times \text{ and } 0 \leq m_i \leq n_i \text{ for all } i.$$
2. Let A be a PID, and $a, b \in A$ elements of the form $a = \prod_{i=1}^k r_i^{n_i}$ and $b = \prod_{j=1}^l s_j^{m_j}$, where $r_i, s_j \in A$ are all irreducible elements, $k, l, m_i, n_j \in \mathbb{Z}_{>0}$, and $r_iA \neq r_{i'}A$ for $i \neq i'$ and $s_jA \neq s_{j'}A$ for $j \neq j'$. Prove that a gcd (defined as in Exercise 1) of a and b is
$$d = \prod_{h=1}^f q_h^{l_h},$$
where
   - $\{q_1, \ldots, q_f\}$ is a finite subset of irreducible elements of A,
   - $q_\alpha A \neq q_\beta A$ for $\alpha \neq \beta$,
   - $\forall h \in \{1, \ldots, f\}$, there exist i, j such that $q_hA = r_iA = s_jA$ and $l_h = \min(m_i, n_j)$.

Please turn over!

3. (*) (Another formulation of the classification of finitely generated torsion modules) Let A be a PID and $M \neq 0$ a finitely generated torsion module. Show that there exists $k \geq 1$ and elements $a_1 | a_2 | \cdots | a_k \in A$ such that $a_i \neq 0$, $a_i \notin A^\times$ for all i and

$$M \cong A/a_1 A \oplus \cdots \oplus A/a_k A.$$

[*Hint:* Use the classification you have seen in class and the Chinese Remainder Theorem]

4. Let G be a finite abelian group generated by two elements.

1. Show that
$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_1 d_2 \mathbb{Z},$$
where $d_1, d_2 \geq 1$ are integers.
2. For every prime p, determine $G(p)$.

5. Let G be a finite abelian group and H be a subgroup of G. Prove: there exists a subgroup $H' \leq G$ such that $H' \cong G/H$. [*Hint:* Abelian groups are $\mathbb{Z}$-modules]

Due to: 27 November 2014, 3 pm.