Exercise sheet 14

[Groups]

1. Let G and H be groups and $\varphi : G \longrightarrow H$ a group homomorphism. If $N \triangleleft G$ is a normal subgroup, and φ is surjective, then show that $\varphi(N) \triangleleft H$.

2. Let $\varphi : G \longrightarrow H$ be a set-theoretic map between groups. Show that φ is a homomorphism if and only if the graph

$$\Gamma_\varphi = \{(x, y) \in G \times H \mid y = \varphi(x)\}$$

is a subgroup of $G \times H$. When is it a normal subgroup?

3. Let G_1 and G_2 be two groups, and let $G = G_1 \times G_2$ be their direct product. Let H be a subgroup of G. We denote by $\pi_i : G \longrightarrow G_i$ the two projection maps to the factors of G, and by $K_i < H$ the kernel of the restriction of π_i to H. We assume that the restrictions of π_1 and π_2 to H are both surjective.

1. Show that π_1 induces by restriction an isomorphism $K_2 \longrightarrow N_1$ where N_1 is a normal subgroup of G_1.
2. Show that if $N_1 = G_1$, then $H = G_1 \times G_2$.
3. Suppose in addition that G_1 and G_2 are simple groups. If $N_1 = \{1\}$, show that $K_1 = \{1\}$ as well. Show in that case that H is the graph of an isomorphism $G_1 \longrightarrow G_2$.

[Rings]

4. Let A be an integral domain and K its fraction field. Show that if B is any ring, then there is a "natural" bijection

$$\{\text{ring morphisms } \psi : K \longrightarrow B\} \longrightarrow \{\text{ring morphisms } \varphi : A \longrightarrow B \text{ such that } \varphi(x) \in B^\times \text{ for all } x \neq 0 \text{ in } A\}.$$

Please turn over!

5. Let A be an integral domain and K its fraction field. Let $I \subset A$ be a *non-zero* prime ideal. Denote

$$A_I = \{x \in K \mid x = a/b \text{ for some } a \text{ and } b \text{ in } A \text{ with } b \notin I\}.$$

1. Show that A_I is a subring of K, and that $A \subset A_I$.
2. Let $J = IA_I$ be the ideal in A_I generated by I. Show that

$$J = \{x \in K \mid x = a/b \text{ for some } a \in I \text{ and some } b \text{ in } A - I\}.$$

3. Show that J is a maximal ideal in A_I, and that it is the unique maximal ideal.
4. Show that the natural ring homomorphism

$$A \longrightarrow A_I/J$$

induces an injective ring homomorphism $A/I \longrightarrow A_I/J$.

6. Let $n \geq 1$ and let A be a real matrix of size $n \times n$ with integral coefficients.

1. Show that

$$\Phi : \begin{cases} \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \\ x \mapsto Ax \end{cases}$$

is a well-defined, $\mathbb{Z}$-linear map.

2. Show that $\ker \Phi$ and $\mathrm{Im}(\Phi)$ are finitely-generated $\mathbb{Z}$-modules. Are they free $\mathbb{Z}$-modules ?
3. Show that $\det(A) \neq 0$ if and only if $\mathrm{Im}(\Phi)$ has finite index in $\mathbb{Z}^n$. Show with an example that Φ is not necessarily surjective.
4. Assume $\det(A) \neq 0$. Try to guess what is the cardinality of the finite set $\mathbb{Z}^n/\,\mathrm{Im}(\Phi)$, as a function of A (and try to prove that this guess is correct...)

[**Fields**]

7. Let K be a field and $L = K(T)$ the field of rational functions with coefficients in K. If $x \in L$ is algebraic over K, show that $x \in K$.

8. Let $K = \mathbb{F}_p$ where p is a prime number and let L/K be a finite extension. Denote by $\varphi : L \longrightarrow L$ the Frobenius morphism.

1. Show that the trace map $\mathrm{tr}_{L/K} : L \longrightarrow K$, as defined in Exercise 1 of Sheet 12, is non-zero (Hint: estimate the size of the kernel of $\mathrm{tr}_{L/K}$.) Deduce that it is surjective.
2. Show also that the norm map $N_{L/K} : L^\times \longrightarrow K^\times$ is surjective.

See next page!

3. Show that

$$\ker(\mathrm{tr}_{L/K}) = \{x \in L \mid x = \varphi(y) - y \text{ for some } y \in L\}$$

and that

$$\ker(N_{L/K}) = \{x \in L \mid x = \frac{\varphi(y)}{y} \text{ for some } y \in L^\times\}.$$

9. Let K be a finite field, $\bar{K}$ an algebraic closure of K. Let $x \in \bar{K}$ be any element, and $P = \mathrm{Irr}(x, K)$ the minimal irreducible polynomial of x in $K[X]$. Let $(x_1, \ldots, x_d)$ be the distinct roots of P in $\bar{K}$. Prove that

$$\prod_{\substack{1 \le i,j \le d \\ i \neq j}} (x_i - x_j)^2 \in K.$$