Exercise sheet 2

The content of the marked exercise **(*)** should be known for the exam.

1. For each of the following groups G and subsets $H \subseteq G$, decide if H is a subgroup of G (in that case, we write $H \leq G$).

1. $G = \mathrm{SL}_2(\mathbb{R})$ and $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$.
2. $G = \mathrm{Sym}(\mathbb{N})$ and $H = \{\sigma \in G : \sigma(n) \neq n \text{ for only finitely many } n \in \mathbb{N}\}$.
3. $G = \mathrm{Sym}(\mathbb{N})$ and $H = \{\sigma \in G : \sigma(n) = n \text{ for only finitely many } n \in \mathbb{N}\}$.
4. G is any group and $H = f^{-1}(H')$, where $f : G \to G'$ is a group homomorphism and H' is a subgroup of G'.
5. $G = \mathrm{Sym}(X)$ and $H = \mathrm{Aut}(X)$, for a fixed group X.
6. G is any group and $H = G_{\mathrm{tor}} := \{g \in G : \exists n \in \mathbb{N}^* : g^n = 1\}$. Prove that $H \leq G$ when G is finite or abelian, but this does not occur when $G = \mathrm{Sym}(\mathbb{N})$.

2. Prove that the following maps are homomorphisms of groups. Find their kernel and image.

1. The absolute value $|\cdot| : \mathbb{C}^\times \to \mathbb{R}^\times$, where $|x + iy| = \sqrt{x^2 + y^2}$ for $x, y \in \mathbb{R}$.
2. $f : \mathbb{R} \to \mathbb{C}^\times$, defined by $f(x) = e^{ix}$.
3. $g : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$, defined by $g(t) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$.

3. Let G be a group and assume that $S \subset G$ is a generating subset for G, i.e. $G = \langle S \rangle$.

1. Assume that $f, g : G \to H$ are two group homomorphisms and that $f(s) = g(s)$ for all $s \in S$. Prove: $f = g$.
2. Assume that $\forall s, t \in S$ we have $st = ts$. Prove that G is abelian.
3. If $s^2 = 1$ for all $s \in S$, does it follow that $x^2 = 1_G$ for all $g \in G$?

Please turn over!

4. Consider the real *Möbius transformations*, that is, the following set of rational functions with coefficients in $\mathbb{R}$:

$$G = \left\{ f(X) = \frac{aX+b}{cX+d} : \ a,b,c,d \in \mathbb{R}, ad-bc \neq 0 \right\},$$

together with the composition of functions $\circ$.

1. Prove that $(G, \circ)$ is a group.
2. Find a subgroup H of G such that $(H, \circ) \cong (\mathbb{R}, +)$ as groups.
3. Consider the map

$$\begin{aligned} \alpha : \mathrm{GL}_2(\mathbb{R}) &\to G \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \frac{aX+b}{cX+d} \end{aligned}$$

   Prove that f is a group homomorphism. Determine its kernel and its image.
4. Determine all Möbius transformations of order 2 (they are also called *involutions*).

5. **(*)** As you have been told in class, Cayley's theorem allows us to embed every group into a symmetric group. Prove it by showing in detail that the following is a well-defined injective group homomorphism:

$$\begin{aligned} \chi : G &\to \mathrm{Sym}(G) \\ g &\mapsto \chi_g : (x \mapsto g \cdot x) \end{aligned}$$

Due to: 2 October 2014, 3 pm.