Exercise sheet 4

The content of the marked exercises **(*)** should be known for the exam.

1. Prove the following two properties of groups:

1. Every subgroup of a cyclic group is cyclic [Recall, we say that a group G is cyclic if $G = \langle g \rangle$ for some $g \in G$. A cyclic group can be either finite or infinite.]
2. Given a group G, if $\mathrm{Aut}(G)$ is cyclic then G is abelian [*Hint:* Consider the conjugation map $G \to \mathrm{Aut}(G)$.]

2. Let H, K be subgroups of G, and assume that $hK = Kh$ for every $h \in H$.

1. Show that:
   - $H \cap K \trianglelefteq H$;
   - $HK \leq G$;
   - $K \trianglelefteq HK$.
2. Prove that there is an isomorphism $H/(H \cap K) \xrightarrow{\sim} HK/K$ [*Hint:* Define first a group homomorphism $H \longrightarrow HK/K$]

3. Let G be a group with a normal subgroup $H \trianglelefteq G$ and consider the canonical projection $\pi : G \to G/H$ sending $g \mapsto gH$. Prove the following statements:

1. If $K \leq G/H$, then $\pi^{-1}(K)$ is a subgroup of G containing H.
2. Conversely, if we have an intermediate subgroup $H \leq K' \leq G$, then $\pi(K') \leq G/H$.
3. The map

$$f : \left\{ \begin{array}{c} \text{subgroups } K', \\ H \leq K' \leq G \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{subgroups} \\ K \leq G/H \end{array} \right\}$$
$$K' \longmapsto \pi(K')$$

   is a bijection.
4. For every $H \leq K' \leq G$, one has that $K' \trianglelefteq G$ if and only if $f(K') \trianglelefteq G/H$.

4. Let G be a group and $H \leq G$ with $[G : H] = 2$. Prove: $H \trianglelefteq G$.

Please turn over!

5. (*) Let A be a simple finite abelian group.

1. Show that A is generated by an element $x \in A$ different from 1_A.
2. Show that $A \cong \mathbb{Z}/k\mathbb{Z}$ where k is a prime. Conversely, show that $\mathbb{Z}/p\mathbb{Z}$ is a simple group for every prime number p.

6. (*) Given two group homomorphisms $\alpha : H \to G$ and $\beta : G \to K$ we say that

$$H \overset{\alpha}{\to} G \overset{\beta}{\to} K$$

is an exact sequence if $\mathrm{Im}(\alpha) = \ker(\beta)$. Moreover, given group morphisms

$$(**) \qquad \cdots \longrightarrow G_{n-2} \overset{\alpha_{n-2}}{\longrightarrow} G_{n-1} \overset{\alpha_{n-1}}{\longrightarrow} G_n \overset{\alpha_n}{\longrightarrow} G_{n+1} \overset{\alpha_{n+1}}{\longrightarrow} G_{n+2} \longrightarrow \cdots$$

we say that $(**)$ is an exact sequence if $G_{i-1} \overset{\alpha_{i-1}}{\longrightarrow} G_i \overset{\alpha_i}{\longrightarrow} G_{i+1}$ is an exact sequence for every i.

We denote by 1 the trivial group $\{1\}$. Notice that for every group G there exists a unique homomorphism $1 \to G$ and a unique homomorphism $G \to 1$.

1. Prove that for any group homomorphism $f : G \to H$ one has:
   - $1 \to G \overset{f}{\to} H$ is an exact sequence if and only if f is injective;
   - $G \overset{f}{\to} H \to 1$ is an exact sequence if and only if f is surjective.
2. We call a short exact sequence any exact sequence of groups of the form

   $$1 \to H \to G \to K \to 1.$$

   Show that given the exact sequence above, there exists a subgroup $H' \trianglelefteq G$ such that $H \cong H'$ and $K \cong G/H'$.

Due to: 16 October 2014, 3 pm.