Commutative Algebra

D-MATH, HS 2014

Prof. Brent R. Doran

Exercise Sheet 1

All exercises are taken from chapter 1 of the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

Let A be a commutative ring with an identity element.

1. Let $A[x]$ be the ring of polynomials in x with coefficients in A. Let $f = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Show that

i) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\iff a_0, \ldots, a_n$ is nilpotent.

iii) f is a zero-divisor $\iff \exists a \neq 0$ in A such that $af = 0$.

iv) f is primitive if the ideal general by $a_0, a_1, \ldots, a_n$ is equal to A, i.e. $(a_0, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff$ f and g are primitive.

2. In the ring $A[x]$, the Jacobson ideal is equal to the nil radical.

3. Let $\mathfrak{a}$ be a proper ideal of A. Show that $\mathfrak{a}$ is radical if and only if it is an intersection of prime ideals of A.

4. Let X be the set of all prime ideals of A. For each subset $E \subset A$, define

$$V(E) = \{\mathfrak{p} \in X \mid E \subset \mathfrak{p}\}.$$

Prove that the sets $V(E)$ satisfy the axioms for closed sets in a topological space, namely prove that:

i) $V(0) = X$ and $V(1) = \varnothing$ where $V(f) := V(\{f\})$.

ii) If $E_i, i \in I$ is any family of subsets of A, then

$$V\Big(\bigcup_{i\in I} E_i\Big) = \bigcap_{i\in I} V(E_i)$$

iii) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b}$ in A.

The resulting topology is called the *Zariski topology.* The set X equipped with the Zariski topology is called the *(prime) spectrum* of A, and is written $\mathrm{Spec}(A)$.

At last, show that if $\mathfrak{a}$ is an ideal in A and $r(\mathfrak{a})$ its radical ideal, then $V(\mathfrak{a}) = V(r(\mathfrak{a}))$

5. For each $f \in A$ let X_f denote the complement of $V(f)$ in $X = \mathrm{Spec}(A)$. The sets X_f are called *basic open sets* of X. Show that

i) The basic open sets form a basis of the Zariski topology.

ii) $X_f \cap X_g = X_{fg}$.

iii) $X_f = \varnothing$ if and only if f is nilpotent.

iv) $X_f = X$ if and only if f is a unit.

v) $X_f = X_g$ if and only if $r((f)) = r((g))$.

vi) X is quasi-compact [1] and that an open subset of X is quasi-compact if and only if it is a finite union of basic open sets.

6. Draw pictures of $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{R})$, $\mathrm{Spec}(\mathbb{C}[x])$, $\mathrm{Spec}(\mathbb{R}[x])$ and $\mathrm{Spec}(\mathbb{Z}[x])$.

7. Let K be a field and let Σ be the set of all irreducibel monic polynomials f in one variable with coefficients in K. Let R be the ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let I be the ideal of R generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that I is proper. Let $\mathfrak{m}$ be a maximal ideal of R containing I, and let $K_1 := R/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L := \bigcup_{n\geq 1} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. Then $\overline{K}$ is an algebraic closure of K.

Due on Tuesday, Sept. 30, 2014

[1] A topological space X is *quasi-compact* if every open cover has a finite subcover.