Commutative Algebra

**D-MATH, HS 2014**

Prof. Brent R. Doran

# Exercise Sheet 10

All exercises are taken from *Introduction to Commutative Algebra* by Atiyah and MacDonald.

Let $A$ denote a ring. We first want to introduce the notion of *direct limit* of a *direct system* of $A$-Modules:

a) *Directed set*: A partially ordered set $I$ is said to be *directed* if for each pair $i, j$ in $I$, there is a $k$ such that $k \geq i$ and $k \geq j$.

b) *Direct system of $A$-Modules*: Let $A$ be a ring, $I$ a directed set and $(M_i)_{i \in I}$ a family of A-modules, so that for each pair $i, j$ with $i \leq j$ there exists an $A$-homomorphism $\mu_{ij} : M_i \longrightarrow M_j$ satisfying the following two conditions:

   i) $\mu_{ii}$ is the identity for all $i \in I$.

   ii) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$.

   Then we call $\mathbf{M} = (M_i, \mu_{ij})$ a *direct system* over the directed set $I$.

c) *Direct limit of* $\mathbf{M}$: Let $C = \oplus_{i \in I} M_i$ be the direct sum of the $M_i$ and identify each module $M_i$ with its canonical image in $C$. Let $D$ be the submodule of $C$ generated by all elements of the form $x_i - \mu_{ij}(x_i)$ for $i \leq j$ and $x_i \in M_i$. Set $M = C/D$ and denote with $\mu_i : M_i \longrightarrow M$ the canonical map (that is, $\mu_i$ is defined as the inclusion $M_i \longrightarrow C$ followed by the projection $C \longrightarrow M$). Then we call the pair consisting of $M$ and the family of homomorphism $\mu_i : M_i \longrightarrow M$ for $i \in I$ the *direct limit* of the direct system $\mathbf{M}$ and denote it by $\varinjlim_{i \in I} M_i$.

**1.** For the direct limit $M$ of a directed system of $A$-Modules $\mathbf{M}$, show that:

   a) $\mu_i = \mu_j \circ \mu_{ij}$.

   b) For every $x \in M$, there is a $i \in I$ and a $x_i \in M_i$ such that $x = \mu_i(x_i)$

   c) If $x_i \in M_i$ such that $\mu_i(x_i) = 0$, then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.

   d) The direct limit is characterized up to isomorphism by the following universal property: For any $A$-module $N$ and for any collection of maps $\alpha_i : M_i \longrightarrow N$, running over all $i \in I$, such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $j \geq i$, there exists a unique homomorphism $\alpha : M \longrightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for any $i \in I$.

**2.** Let $\mathbf{M} = (M_i, \mu_{ij})$ be a directed system of $A$-Modules and let $M$ be the direct limit. We want to give an alternative description of the direct limit of $\mathbf{M}$. Define the set

$$C := \coprod_{i \in I} M_i / \sim$$

where $(x_i, i) \sim (x_j, j)$ if and only if there is a $k \in I$ with $k \geq i$ and $k \geq j$ so that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Define addition $+$ as follows: For $a = [(x_i, i)]$ and $b = [(x_j, j)]$ in $C$, let $k \in I$ so that $k \geq i, j$ and set $a + b = [(\mu_{ik}(x_i) + \mu_{jk}(x_j), k)]$.

a) Show that $\sim$ is an equivalence relation.

b) Prove that the addition operation $+$ is well defined.

c) Define a multiplication operation $\cdot : A \times C \longrightarrow C$ and show that it is well defined.

d) Prove that $(C, +, \cdot)$ is an $A$-Module.

e) Finally show that $C$ is isomorphic to the direct limit $M$.

**3.** Let $A$ be the ring of germs at 0 of $C^\infty$-functions on $\mathbb{R}$ and let $\mathfrak{m} = (x)$ denote the ideal generated by the coordinate function $x$. It is the unique maximal ideal of $R$. Show by elementary calculus that if $f$ is a $C^\infty$-function such that all its derivatives vanish at the origin, then $f/x$ is also such a function. Conclude that $\mathfrak{m}(\bigcap_{j \geq 1} \mathfrak{m}^j) = \bigcap_{j \geq 1} \mathfrak{m}^j$. Moreover, recall that the intersection $\bigcap_{j \geq 1} \mathfrak{m}^j$ is non-zero. It contains for instance $e^{-\frac{1}{x^2}}$ (see Chapter 5.3 in Eisenbud). This example therefore shows that the finiteness conditions are necessary in both Nakayama's Lemma and Krull's Intersection Theorem.

**4.** Let $\mathfrak{a}$ be an ideal of an noetherian ring $A$ and let $M$ be a finitely generated $A$-module. Show that there is a largest submodule $N$ of $M$ with the property that $N$ is annihilated by an element of the form $1 - a$ with $a \in \mathfrak{a}$. Moreover, show $\bigcap_{j \geq 1} \mathfrak{a}^j M = N$. Understand this as a converse to Krull's Intersection Theorem.

**5.** Let $\mathfrak{a}$ be an ideal of a noetherian ring $A$ and let $M$ be a finitely generated $A$-module. Show that

$$\bigcap_{n \geq 1} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}}),$$

where $\mathfrak{m}$ runs over all maximal ideals containing $\mathfrak{a}$. Deduce that

$$\hat{M} = 0 \Leftrightarrow \operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset.$$

**6.** Let $M$ be a non-zero finitely generated torsion module over a Dedekind domain $A$. Show that $M$ is uniquely representable as a finite direct sum of modules $A/\mathfrak{p}_i^{n_i}$, where the $\mathfrak{p}_i$ are non-zero prime ideals of $A$. [Hint: Use the corresponding fact for modules over principal ideal domains.]

**Due on Tuesday, 2.12. 2014**