Commutative Algebra

D-MATH, HS 2014

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Exercise Sheet 11

Exercises 3. - 6. are taken from Chapter 10 of *Introduction to Commutative Algebra* by Atiyah and MacDonald.

1. Let A be a ring and let $X = \operatorname{Spec}(A)$ be its spectrum. Recall that to any $f \in A$ we have associated a basic open set X_f (see Exercise Sheet 1). Show that if $f, g \in A$, then

$$X_g \subset X_f \quad \Longleftrightarrow \quad \exists u \in A\ \exists n \geq 1 \text{ such that } g^n = hf.$$

Conclude the following facts:

(i) If $U = X_f$ for some $f \in A$, then the ring $A(U) := A_f$ depends only on U and not on f. Here A_f is the localization of A along f.

(ii) If $U' = X_g$ is another basic open set such that $X_g \subset X_f$, define a homomorphism $\rho = \rho_{U'U} : A(U) \to A(U')$. We call ρ the restriction homomorphism.

(iii) The restriction homomorphisms satisfy $\rho_{UU} = id$ and for every inclusion $U'' \subset U' \subset U$ of basic open sets, we have $\rho_{U''U} = \rho_{U''U'} \circ \rho_{U'U}$.

(iv) Let $x = \mathfrak{p}$ be a point of X. Then

$$\varinjlim_{x \in U} A(U) \cong A_{\mathfrak{p}}$$

Here the direct limit is taken over the directed system $(\{A(U)\}_U, \{\rho_{U'U}\}_{U'U})$ where the index set runs over all basic open sets U that contain x and the ordering is given by inclusion.

The assignment of the ring $A(U)$ for every basic open set U and the restriction homomorphisms $\rho_{U'U}$ for every pair $U' \subset U$ of basic open sets, that satisfy (iii) above, defines a *presheaf of rings* on the basis of open sets $(X_f)_{f \in A}$ of X. (iv) shows that the stalk of this presheaf at $x \in X$ is the corresponding local ring $A_{\mathfrak{p}}$.

2. In the notation from the previous exercise, let $(U_i)_{i \in I}$ be a covering of X by basic open sets. Prove:
If $s_i \in A(U_i)\ \forall i \in I$ is given such that for every i, j the images of s_i and s_j in $A(U_i \cap U_j)$ are equal (i.e. such that $\rho_{U_i \cap U_j, U_i}(s_i) = \rho_{U_i \cap U_j, U_j}(s_j) \forall i, j$), then there exists a unique $s \in A = A(X)$ such that $\rho_{UX}(s) = s_i$ for all $i \in I$. Essentially, this implies that the presheaf is a *sheaf*.

3. Let p be a prime number. Consider for any $n \geq 1$ the injection of abelian groups $\alpha_n : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ given by $\alpha_n(1) = p^{n-1}$. Let $\alpha : A \to B$ be the direct sum of all the α_n, where A is a countable direct sum of the $\mathbb{Z}/p\mathbb{Z}$ and B is the countable direct sum of the $\mathbb{Z}/p^n\mathbb{Z}$. Show that the p-adic completion of A is A and that the completion of A for the topology induced from the p-adic topology on B is the direct product of the $\mathbb{Z}/p\mathbb{Z}$. Deduce that p-adic completion is not a right-exact functor on the category of all $\mathbb{Z}$-modules.

4. Recall that the $\mathfrak{a}$-adic completion of a noetherian ring A with respect to an ideal $\mathfrak{a}$ of A is noetherian. The converse however is not true. Consider the ring A of germs at 0 of C^∞-functions on $\mathbb{R}$. We have already seen in Sheet 10 that it is not noetherian. Show that, however, its completion with respect to its unique maximal ideal is noetherian. (Hint: Use Borel's theorem that any power series occurs as the Taylor expansion of some C^∞-function.)

5. Let A be a local ring with maximal ideal $\mathfrak{m}$. Assume that A is $\mathfrak{m}$-adically complete. For any polynomial $f(x) \in A[x]$, let $\overline{f}(x) \in (A/\mathfrak{m})[x]$ denote its reduction modulo $\mathfrak{m}$. Prove *Hensel's Lemma*: If $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\overline{g}(x), \overline{h}(x) \in (A/\mathfrak{m})[x]$ of degrees $r, n-r$ with $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, then we can lift $\overline{g}$ and $\overline{h}$ back to monic polynomials $g(x)$ and $h(x)$ in $A[x]$ with $f(x) = g(x)h(x)$. (For hints see Exercise 9 in Chapter 10 of A-M.)

Deduce that if $\overline{f}(x)$ has a simple root $\alpha \in A/\mathfrak{m}$, then $f(x)$ has a simple root $a \in A$ with $a = \alpha$ modulo $\mathfrak{m}$.

6. Apply Hensel's Lemma in the following two exercises.

(i) Show that 2 is a square in the ring of 7-adic integers.

(ii) Let k be a field and let $f(x, y) \in k[x, y]$. Suppose that the polynomial $f(0, y) \in k[y]$ has a simple root. Prove that there exists a formal power series $y(x) \in k[[x]]$ such that $f(x, y(x)) = 0$. View this as an implicite function theorem.

Due on Tuesday, 9.12. 2014