Commutative Algebra

D-MATH, HS 2014

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Exercise Sheet 2

All exercises are taken from chapter 1 and 2 of the book *Introduction to Commutative Algebra* by Atiyah and Macdonald.

Let A be a commutative ring with a unit element.

1. A topological space X is *irreducible* if $X \neq \varnothing$ and if every two non-empty open sets in X intersect. This is equivalent to the statement that every non-empty open set is dense in X. Consider a radical ideal $\mathfrak{a}$ of A and the associated $V(\mathfrak{a}) \subset \operatorname{Spec}(A)$ endowed with the subset topology. Show that $V(\mathfrak{a})$ is irreducible if and only if $\mathfrak{a}$ is a prime ideal and that, in particular, $\operatorname{Spec}(A)$ is irreducible if and only if the nilradical of A is prime.

2. Let X be a topological space. Show:

(a) If Y is an irreducible subspace of X, then the closure of $\overline{Y}$ of Y in X is irreducible.

(b) Every irreducible subspace of X is contained in a maximal irreducible subspace.

(c) The maximal irreducible subspaces of X are closed and cover X. They are called the *irreducible components* of X. What are the irreducible components of a Hausdorff space?

(d) If $X = \operatorname{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

3. Let $\varphi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\varphi^{-1}(\mathfrak{q})$ is an element of X, hence φ induces a map $\varphi^* : Y \to X$.

(a) If $f \in A$ then $(\varphi^*)^{-1}(X_f) = Y_{\varphi(f)}$. Conclude that φ^* is continuous.

(b) Show that $\operatorname{Spec}(-)$ induces a contravariant functor from the category of commutative rings to the category of topological spaces.

(c) If $\mathfrak{a}$ is an ideal of A, then $(\varphi^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(d) If $\mathfrak{b}$ is an ideal of B, then $\overline{\varphi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

(e) If φ is surjective, then φ^* is a homeomorphism of Y onto the closed subset $V(\operatorname{Ker}(\varphi))$ of X. (In particular, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\mathfrak{N})$ are naturally isomorphic, where $\mathfrak{N}$ is the nilradical of A.)

(f) If φ is injective, then $\varphi^*(Y)$ is dense in X. More precisely, $\varphi^*(Y)$ is dense in X if and only if $\operatorname{Ker}(\varphi) \subseteq \mathfrak{N}$.

(g) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\varphi : A \to B$ by $\varphi(x) = (\overline{x}, x)$, where $\overline{x}$ is the image of x in $A/\mathfrak{p}$. Show that φ^* is bijective but not a homeomorphism.

4. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$ for any integers n, m.

5. Let $\mathfrak{a}$ be an ideal of A and let M be an A-module. Show that $(A/\mathfrak{a}) \otimes_A M \cong (M/\mathfrak{a}M)$.

6. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Show that M is finitely generated if M' and M'' are.

Due on Tuesday, 7.10. 2014