Commutative Algebra

D-MATH, HS 2014

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Exercise Sheet 5

All exercises are taken from chapter 4 of the book *Introduction to Commutative Algebra* by Atiyah and MacDonald.

Let A be a non-zero commutative ring with a unit element.

1. If an ideal $\mathfrak{a} \subset A$ has a primary decomposition, then $\mathrm{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

2. Let k be a field and let $k[x, y, z]$ be the polynomial ring in the independent indeterminates x, y, z. Consider the ideals $\mathfrak{p}_1 := (x, y)$, $\mathfrak{p}_2 := (x, z)$ and $\mathfrak{m} := (x, y, z)$. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of the ideal $\mathfrak{p}_1\mathfrak{p}_2$. Which components are isolated and which are embedded?

3. Let $A[x]$ denote the ring of polynomials in one indeterminate over A. For each ideal $\mathfrak{a} \subset A$, let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in $\mathfrak{a}$. Show that

(i) $\mathfrak{a}[x]$ is the extension of $\mathfrak{a}$ to $A[x]$.

(ii) If $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

(iii) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$. [Hint: Use Exercise 1 of the first exercise sheet.]

(iv) If $\mathfrak{a} = \bigcap_{1\leq i\leq n} \mathfrak{q}_i$ is a minimal primary decomposition in A, then $\mathfrak{a}[x] = \bigcap_{1\leq i\leq n} \mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.

(v) If $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$

4. Let k be a field. Show that in the polynomial ring $k[x_1, \ldots, x_n]$ with n independent indeterminates the ideals $(x_1, \ldots, x_i)$, where $1 \leq i \leq n$, are prime and all their powers are primary. (Hint: Use Exercise 3.)

5. Let $\mathfrak{p}$ be a prime ideal of A. For any positive natural number n the *nth symbolic power of* $\mathfrak{p}$ is defined to be the contraction of the extension of $\mathfrak{p}^n$ under the localization homomorphism $A \to A_{\mathfrak{p}}$. We denote it by $\mathfrak{p}^{(n)}$. Show that:

(i) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.

(ii) If $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component.

(iii) We have $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary.

6. Let $\mathfrak{p}$ be a prime ideal of A and denote by $S_{\mathfrak{p}}(0)$ the kernel of the localization homomorphism $A \to A_{\mathfrak{p}}$. Show that every $\mathfrak{p}$-primary ideal of A contains $S_{\mathfrak{p}}(0)$. Suppose moreover that A satisfies the following condition: For any prime ideal $\mathfrak{p}$ of A the intersection of all $\mathfrak{p}$-primary ideals equals $S_{\mathfrak{p}}(0)$ (This condition is satisfied by any Noetherian ring). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals, none of which is a minimal prime ideal of A. Show that there exists an ideal $\mathfrak{a} \subset A$ whose associated prime ideals are $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. [To get hints see Exercise 19 in Chapter 4 of Atiyah MacDonald's book.]

Due on Tuesday, 28.10. 2014