Commutative Algebra

D-MATH, HS 2014

Prof. Brent R. Doran

Exercise Sheet 7

Exercises 1 to 5 are taken from Chapter 6 and 7 of *Introduction to Commutative Algebra* by Atiyah and MacDonald.

Let X be a topological space. X is said to be *noetherian* if the open sets in X satisfy the ascending chain condition, i.e. that any chain

$$U_1 \subset U_2 \subset U_3 \subset \dots$$

of increasing open subsets of X becomes stationary. Equivalently, the set of closed subsets satisfy the descending chain condition.

1. Prove that the following conditions are equivalent.

(i) X is noetherian

(i) Every subspace of X is noetherian

(iii) Every open subspace of X is quasi-compact

(iv) Every subspace of X is quasi-compact.

Moreover, show that a noetherian space is a finite union of irreducible closed subspaces. (*Hint:* Let Σ be the set of closed subsets of X, which are not a finite union of irreducible closed subspaces. If we assume that Σ is non-empty, then there exists a minimal element.) Conclude that the set of irreducible components of a noetherian space is finite.

2. Let A be a ring. Show that if A is noetherian, then $\operatorname{Spec}(A)$ noetherian. Conclude that the set of minimal prime ideals of any noetherian ring is finite. Is it also true that if $\operatorname{Spec}(A)$ is noetherian, then A is noetherian? (*Hint*: Consider $k[x_1, x_2, x_3, \dots]/(x_1, x_2^2, x_3^3, \dots)$.)

3. Let A be a ring and M be a noetherian A-module. Consider any surjective A-module homomorphism $u : M \to M$. Show that u is an isomorphism. (*Hint*: Consider $\operatorname{Ker}(u^n)$ for $n \geq 1$.)

4. Let $A[[x]]$ be the ring of power series in one variable over a ring A. Show that if A is noetherian, then $A[[x]]$ is noetherian. Is the converse also true?

5. Which of the following rings are noetherian?

(i) The ring of rational functions of z having no pole on the circle $|z| = 1$.

(ii) The ring of power series in z with a positive radius of convergence.

(iii) The ring of power series in z with an infinite radius of convergence.

(iv) The ring of polynomials in z whose first k derivatives vanish at the origin, where k is a fixed non-negative integer.

(v) The ring of polynomials in z, w whose partial derivatives with respect to w vanish for $z = 0$.

In all cases the coefficients are complex numbers.

6. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{C}$ is not noetherian. (*Hint*: Fix a prime number p and consider 2^n-th roots of p in $\mathbb{C}$ for all $n \geq 1$.)

Due on Tuesday, 11.11. 2014