Commutative Algebra
D-MATH, HS 2014

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Exercise Sheet 8

All exercises are taken from Chapter 5 and 8 of *Introduction to Commutative Algebra* by Atiyah and MacDonald.

1. Let A be a Noetherian ring. Show that the following are equivalent:

i) A is Artinian.

ii) $\operatorname{Spec}(A)$ is discrete and finite.

iii) $\operatorname{Spec}(A)$ is discrete.

2. Let k be a field and A a finitely generated k-algebra. Show that the following are equivalent:

i) A is Artinian.

ii) A is a finite k-algebra.

(*Hint*: For i) $\to$ ii), use the fact that every Artinian ring is a finite product of local Artinian rings to reduce to the case of a Artin local ring. By the Nullstellensatz the residue field is a finite extension of k. Finally use that the length of A as an A-module is finite. For ii) $\to$ i), use the descending chain condition for the ideals of A, which are k-vector spaces.)

3. Consider a ring homomorphism $f : A \to B$. The main goal of this exercise is to show that if f is finite, then the fibres of f^* are finite. More precisely, consider the following statements:

(i) f is finite.

(ii) For each prime ideal $\mathfrak{p}$ of A the ring $B_{\mathfrak{p}} \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra, where $k(\mathfrak{p})$ denotes the residue field of $A_{\mathfrak{p}}$.

(iii) Each fibre of f^* is finite.

Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). (Hint: Use Exercise 21 of Chapter 3 in A-M.) Moreover, recall that f is finite if and only if it is of finite type and integral (see e.g. Chapter 5 in A-M). Show that

(iv) there exists a ring homomorphism which is integral and has finite fibers, but which is not finite.

The remaining exercises deal with two characterizations of the concept of valuation rings.

4. Let A and B be two local rings with $A \subset B$. We say that B *dominates* A if the maximal ideal $\mathfrak{m}$ of A is contained in the maximal ideal $\mathfrak{n}$ of B, or equivalently $\mathfrak{m} = \mathfrak{n} \cap A$.

Let K be a field. Consider the set Σ of all local subrings of K. We define an ordering on Σ by the relation of domination. Show that the maximal elements of Σ are precisely the valuation rings of K.

5. Let A be a valuation ring of a field K and denote with U the set of units of A. U is naturally a subgroup of the multiplicative group K^* of the field K.

Let $\Gamma = K^*/U$ be the quotient group. If $\xi, \eta \in \Gamma$ are represented by $x, y \in K$, define

$$\xi \geq \eta \qquad \Longleftrightarrow \qquad xy^{-1} \in A$$

Show:

(i) $\geq$ defines a total ordering on Γ.

(ii) $\geq$ is compatible with the group structure on Γ, i.e. show that if $\xi \geq \eta$ and $\omega \in \Gamma$, then $\xi\omega \geq \eta\omega$.

(iii) Let $\nu : K^* \to \Gamma$ be the projection. Show that for all $x, y \in K^*$ we have $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

Properties (i) and (ii) show that Γ is a totally ordered abelian group. It is called the *value group* of A. The map ν is called a *valuation* of the field K with value group Γ.

6. Show conversely that every valuation of a field K is induced by a valuation ring. More precisely, let Γ be a totally ordered abelian group, with group operation $+$ and let K be a field. A *valuation* of K with values in Γ is a map $\nu : K^* \to \Gamma$ such that for all $x, y \in K^*$

(i) $\nu(xy) = \nu(x) + \nu(y)$

(ii) $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

Show that $A := \{x \in K^* : \nu(x) \geq 0\} \cup \{0\}$ is a valuation ring of K. It is called the *valuation ring* of ν and the subgroup $\nu(K^*)$ of Γ is the *value group* of ν.

Due on Tuesday, 18.11. 2014