Commutative Algebra

D-MATH, HS 2014

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Exercise Sheet 9

All exercises are taken from *Introduction to Commutative Algebra* by Atiyah and MacDonald.

1. Let k be an algebraically closed field. Using Hilberts Nullstellensatz (i.e. Corollary 5.24 in A-M), show that the maximal ideals of $A := k[x_1, \ldots, x_n]$ are precisely the ideals of the form $(x_1 - a_1, \ldots, x_n - a_n)$, where $(a_1, \ldots a_n) \in k^n$. In particular, any proper ideal of A vanishes at least at one point of k^n.

Let $V \subset k^n$ be an affine algebraic variety defined by an ideal $\mathfrak{a} \subset A$. Show moreover that $I(V) = \sqrt{\mathfrak{a}}$.

2. Let A be an integral domain and K its field of fractions. Show that the following are equivalent:

(i) A is a valuation ring of K.

(ii) For any ideals $\mathfrak{a}, \mathfrak{b}$ of A, we have $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$.

Using this equivalence show that if A is a valuation ring and $\mathfrak{p}$ a prime ideal of A, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are valuation rings of their fields of fractions.

3. Let A be a valuation ring of a field K. Show that every subring B of K that contains A is a local ring of A, i.e. there is a prime ideal $\mathfrak{p}$ of A such that $B = A_{\mathfrak{p}}$.

4. Let A be a ring such that its localization at any prime ideal is noetherian. Is A necessarily noetherian?

5. Show that a valuation ring is noetherian if and only if it is a field or a discrete valuation ring.

6. Let A be a Dedekind domain, S a multiplicatively closed subset of A. Show that $S^{-1}A$ is either a Dedekind domain or the field of fractions of A.

Suppose that $S \neq A \setminus \{0\}$, and let H, H' be the ideal class groups of A and $S^{-1}A$ respectively. Show that extension of ideals induces a surjective ring homomorphism $H \to H'$.

Due on Tuesday, 25.11. 2014