Commutative Algebra

D-MATH, HS 2014

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Solutions Sheet 1

1.

Lemma 1. *Let $a \in A$ be a unit and $b \in A$ be nilpotent. Then $a+b$ is a unit.*

Proof. Let n be a positive natural number such that $b^n = 0$. Then $a+b$ has inverse $a^{-1}(1 + \sum_{1 \le k \le n-1} (-a^{-1}b)^k)$. □

(i) One direction is an immediate consequence of the Lemma.
So it remains to show that if f is a unit, then $a_1, a_2, \ldots, a_n$ are nilpotent. Let $g = b_0 + b_1 x + \cdots + b_m x^m$ be the inverse of f. Looking at the degree zero terms of

$$(a_0 + a_1 x + \cdots + a_n x^n)(b_0 + b_1 x + \cdots + b_m x^m) = 1,$$

we observe $b_0 a_0 = 1$. Hence b_0 is a unit.
We prove by induction on r that $a_n^{r+1} b_{m-r} = 0$. Looking at the highest degree term of fg, we get $a_n b_m = 0$. Suppose it is proved for $k \le r-1$. We have $a_{n-r} b_m + a_{n-r+1} b_{m-1} + \cdots + a_n b_{m-r} = 0$ by comparing the coefficients of x^{m+n-r}. Hence

$$a_n^{r+1} b_{m-r} = -a_n^r (a_{n-r} b_m + a_{n-r+1} b_{m-1} + \cdots + a_{n-1} b_{m-r+1}) = 0.$$

Finally for $r = m$ our result $a_n^{m+1} b_0 = 0$ shows that a_n is nilpotent, as b_0 is a unit. By the Lemma, also $f - a_n x^n$ is a unit. We can therefore proceed recursively.

(ii) Suppose $f^k = 0$. This implies $a_0^k = 0$. Let $h = 1 + f$. By the Lemma we know that h is a unit. Therefore $a_1, \cdots, a_n$ are all nilpotent as we just proved above. The other direction follows from the fact that the nilpotent elements of A form an ideal.

(iii) We prove that if f is a zero-divisor, then there exists a non-zero a in A such that $af = 0$.
Suppose $g = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m \neq 0$ such that $fg = 0$. We may assume that m is the minimal natural number for which such a polynomial exists. It is easy to see that $a_n b_m = 0$. Hence $a_n g = 0$ because $a_n g$ annihilates f and has degree $< m$. Now we use induction on $0 \le r \le n$ to show that $a_{n-r} g = 0$: Assuming it is true for $k \le r-1$, we have $g(a_0 + \cdots + a_{n-r} x^{n-r}) = 0$. As above, $a_{n-r} b_m = 0$, which implies that $a_{n-r} g = 0$ by minimality of m.
Hence $a_i b_j = 0$ for $0 \le i \le n$ and $0 \le j \le m$ and therefore each b_j annihilates f.

(iv) For any $f \in A[X]$ we write $\langle f \rangle$ for the ideal of A generated by the coefficients of f.

As $\langle fg \rangle \subset \langle f \rangle \cap \langle g \rangle$ we see that if fg is primitive, then both f and g are so.

Assume conversly that $\langle fg \rangle \neq (1)$. Then there exists a maximal ideal $\mathfrak{m}$ containing $\langle fg \rangle$. Define k to be the field $A/\mathfrak{m}$. We have a homomorphism $A[X] \to k[X]$, $f \mapsto \overline{f}$ by reducing coefficients modulo $\mathfrak{m}$. So $\overline{f}\overline{g} = \overline{fg} = 0$ in the integral domain $k[x]$. Hence either $\overline{f} = 0$ or $\overline{g} = 0$. This means all coefficients of f or of g are contained in $\mathfrak{m}$ and so f or g cannot be primitive.

2. By definition the nilradical $\mathfrak{N}$, respectively the Jacobson ideal $\mathfrak{J}$, of $A[x]$ is the intersection of all prime ideals, respectively of all maximal ideals, of $A[x]$. Therefore $\mathfrak{N} \subset \mathfrak{J}$. To see $\mathfrak{J} \subset \mathfrak{N}$ let $g(x) = g_n x^n + \cdots + g_0 \in \mathfrak{J}$. Then $1 + g \cdot h$ is a unit for any $h \in A[x]$ (see for instance [A.-M., Prop 1.9]) and in particular for $h(x) = x$. Applying the first part of Exercise 1 to $1 + g(x) \cdot x$ we thus see that g_i is nilpotent for any $0 \leq i \leq n$. By the second part of Exercise 1 this is equivalent to $g \in \mathfrak{N}$.

3. We use the fact that for any ideal $\mathfrak{b} \subset A$ its radical is the intersection of all prime ideals which contain $\mathfrak{b}$ (see for instance [A-M, Prop. 1.14]). This implies that any radical ideal is an intersection of prime ideals. Conversely, by the mentioned fact, if $\mathfrak{a} \subset A$ is an intersection of prime ideals, we see that it contains its radical and therefore is radical.

4. (i) Any prime ideal contains 0, but not 1.

(ii) This follows immediately from the definition.

(iii) The inclusions $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a}, \mathfrak{b}$ induce an inclusion $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{ab})$. Take any prime ideal $\mathfrak{p}$ containing $\mathfrak{ab}$. Without loss of generality we may assume $\mathfrak{a} \not\subset \mathfrak{p}$ and hence an element $a \in \mathfrak{a} \backslash \mathfrak{p}$. Then $a \cdot \mathfrak{b} \subset \mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{p}$. As $\mathfrak{p}$ is prime, this implies $\mathfrak{b} \subset \mathfrak{p}$. We therefore also see $V(\mathfrak{ab}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(iv) For any positive integer k and any $f \in A$, a prime ideal contains f^k if and only if it contains f.

5. (i) Any open set is of the form

$$X_I = \{\mathfrak{p} \in \mathrm{Spec} A \mid I \not\subset \mathfrak{p}\}$$

for $I \subset A$. Take any $\mathfrak{p} \in X_I$. Hence there is $f \in I$ such that $f \notin \mathfrak{p}$. We see $\mathfrak{p} \in X_f \subset X_I$. Hence the basic open subset form a basis.

(ii) A prime ideal $\mathfrak{p}$ does not contain fg if and only if it contains neither f nor g.

(iii) This is the statement [A-M, Prop. 1.8] that

$$\mathrm{nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p}.$$

(iv) An element $x \in A$ is a unit if and only if it is not contained in any prime ideal.

(v) Equivalently, $V((f)) = V((g))$ if and only if $r((f)) = r((g))$. This we conclude using

$$\operatorname{rad}(I) = \bigcap_{I \subset \mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$$

and applying [A-M, Prop. 1.8] to the ring A/I.

(vi) Take any open covering of $\operatorname{Spec} A$. Refine it to an open covering

$$\bigcup_{i \in I} X_{f_i} = \operatorname{Spec} A$$

by basic open subsets. By taking complements we get the equality

$$\bigcap_{i \in I} V((f_i)) = V((\{f_i\}_{i \in I})) = \emptyset.$$

Hence $(\{f_i\}_{i \in I}) = A$. So there is a finite subset $J \subset I$ such that $1 = \sum_{j \in J} g_j f_j$ for some $g_j \in A$. Hence the X_{f_j} for $j \in J$ form a finite subcover.

Any compact open subset can be covered by finitely many basic open subsets as they are a basis. We remark that $X_f \simeq \operatorname{Spec} A[x]/(xf - 1)$. This can be seen directly. We will however later get to know the concept of localization and treat this isomorphism in a general version. Hence a basic open subset is compact. Thus also finite unions of them are compact.

6. We only determine, without proof, the prime ideals of $\mathbb{Z}[x]$. These are precisely the ideals of the form

(a) (0),

(b) (p), for some prime number $p \in \mathbb{Z}$,

(c) (f), for some irreducible polynomial $f \in \mathbb{Z}[x]$, or

(d) (p, f), for some prime number $p \in \mathbb{Z}$ and some polynomial $f \in \mathbb{Z}[x]$ which is irreducible modulo p.

7. We show that I is a proper ideal: Assume by contradiction that $I = R$. We therefore find irreducible monic polynomials $f_1, \ldots, f_n$ such that $f_1(x_{f_1}), \ldots, f_n(x_{f_n})$ generate the unit ideal in $A_n := K[x_{f_1}, \ldots, x_{f_n}]$. We may assume that n is the minimal natural number for which such polynomials exist. Let $A_{n-1} := K[x_{f_1}, \ldots, x_{f_{n-1}}]$ and let $J \subset A_{n-1}$ be the ideal generated by $f_1(x_{f_1}), \ldots, f_{n-1}(x_{f_{n-1}})$. By minimality of n, the ideal J is proper, i.e. $A_{n-1}/J \neq 0$. We get a ring homomorphism

$$A_n = A_{n-1}[x_{f_n}] \xrightarrow{\phi} (A_{n-1}/J)[x_{f_n}]$$

through reducing coefficients modulo I. Let $\overline{f}_n(x_{f_n}) := \phi(f_n(x_{f_n}))$. This has degree greater or equal to 1 as f_n is irreducible. It does therefore not generate the unit ideal. Hence also $\phi^{-1}((\overline{f}_n(x_{f_n}))) \subsetneq A_n$. But $f_1(x_{f_1}), \ldots, f_n(x_{f_n}) \in \phi^{-1}((\overline{f}_n(x_{f_n})))$ which contradicts the assumption, that these elements generate A_n.