Solutions Sheet 10

1. (a) This is immediate from the definition.

(b) Let x be represented by an element $c = \Sigma_{i \in J} x_j \in C$, where J is a finite subset of I and $x_j \in M_j$ for any $j \in J$. As I is directed, there exists $i \in I$ such that $j \leq i$ for any $j \in J$. For $x_i := \Sigma_{j \in J} \mu_{ji}(x_j) \in M_i$ it is immediately checked that $\mu_i(x_i) = x$.

(c) If $\mu(x_i) = 0$, then $x_i = \sum_{(k,j)} x_k - \mu_{kj}(x_k) =: S$, where the sum S runs over a finite set of tuples $(j, k) \in I^2$ with $j \leq k$ and $x_k \in M_k$. Let $l \in I$ be such that l is greater than any of these k and j. We claim that $\mu_{il}(x_i) = \sum_{(k,j)} \mu_{kl}(x_k) - \mu_{jl}(\mu_{kj}(x_k))$. By the compatibility condition $\mu_{kl} = \mu_{jl} \circ \mu_{kj}$, this will imply that $\mu_{il}(x_i) = 0$. To see the claim, consider for any $t \in I$ the components occurring in the sum S which lie in M_t and write down an explicit expression for their sum which we denote by S_t. We have thus split S into the sum $S = \sum_t S_t$. We may only look at finitely many S_t, namely at those for which t occurs in one of the (k, j). Moreover for any $t \neq i$ we have $S_t = 0$. Hence $\mu_{il}(x_i) = \mu_{il}(S_i) = \sum_t \mu_{tl}(S_t) = \sum_{(k,j)} \mu_{kl}(x_k) - \mu_{jl}(\mu_{kj}(x_k))$.

(d) By part b) the map α is uniquely determined, namely by $\alpha(x) = \alpha_i(x_i)$, whenever $x = \mu_i(x_i)$ for some $i \in I$. Using the various compatibility conditions it is straightforwardly checked that defining $\alpha(x)$ in this way is unambiguous and indeed yields a homomorphism with the desired properties.

2. Parts a) to d) are straightforward exercises. For part e) note that it suffices to prove that C together with the canonical homomorphisms $\mu_i : M_i \to C$, which are inclusion in $\coprod M_i$ followed by the quotient map to C, satisfies the universal property described in Exercise 1, d). This is again a straightforward check.

3. We only sketch the proof, which is purely analytic. Firstly, one can show by induction on k and using Bernoulli de l'Hopital that for any such f and any $k \geq 0$ the first derivative of f/x^k at 0 exists and equals 0. Secondly, one can show that the same is true for any j-th derivative, where $j \geq 1$, which can be done by induction on j using the first part. This implies $f/x \in \mathfrak{m}^j$ for any $j \geq 1$ and consequently $f \in \mathfrak{m}(\bigcap_{j \geq 1} \mathfrak{m}^j)$. As all of the derivatives at 0 of any $f \in \bigcap_{j \geq 1} \mathfrak{m}^j$ vanish, we deduce $\bigcap_{j \geq 1} \mathfrak{m}^j \subset \mathfrak{m}(\bigcap_{j \geq 1} \mathfrak{m}^j)$.

4. By Krull's intersection theorem we only need to show that any A-submodule N of M for which there exists $a \in \mathfrak{a}$ with $(1-a)N = 0$ is contained in $\bigcap_{j\geq 1} \mathfrak{a}^j M$. Let $n \in N$. We have $(1-a)n = 0$, i.e $n = an$, and thus inductively get $n = a^j n \in \mathfrak{a}^j M$ for any $j \geq 1$ as needed.

5. We set $N := \bigcap_{j\geq 1} \mathfrak{a}^j M$ and $N' := \bigcap_{\mathfrak{m}\supset\mathfrak{a}} \operatorname{Ker}(M \to M_{\mathfrak{m}})$. By Krull's Intersection Theorem there exists $a \in \mathfrak{a}$ with $(1+a)n = 0$ for all $n \in N$. As $1+a \notin \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \supset \mathfrak{a}$ we get $N \subset N'$. On the other hand, by definition we have $N'_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \supset \mathfrak{a}$ or, equivalently, $(N'/\mathfrak{a}N')_{\overline{\mathfrak{m}}} = 0$ for all maximal ideals $\overline{\mathfrak{m}}$ of $A/\mathfrak{a}$. As the vanishing of a module is a local property (see Proposition 3.8 in A-M), we have $N' = \mathfrak{a}N'$ and, by Nakayama's Lemma, thus obtain $a' \in \mathfrak{a}$ with $(1+a')N' = 0$. Exercise 4 then implies $N' \subset N$.

We deduce that

$$\hat{M} = 0 \Leftrightarrow M = N \Leftrightarrow M = N' \Leftrightarrow \operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset.$$

6. By assumption on M, the annihilator $\mathfrak{a} := \operatorname{Ann}(M)$ of M is a non-zero proper ideal of A. The quotient $A/\mathfrak{a}$ is therefore an artin ring and thus uniquely a finite product $\prod_{1\leq i\leq n} A_i$ of artin local rings $A_i = A/(\mathfrak{m}_i^{n_i})$, where $\mathfrak{m}_i$ are maximal ideals of A containing $\mathfrak{a}$ (see Proposition 8.7 in A-M.). More precisely, n, the $\mathfrak{m}_i$ and the n_i are uniquely determined. We may view M as an $A/\mathfrak{a}$-module and therefore get a decomposition $M = \bigoplus_{1\leq i\leq n} M_i$ with A_i-modules M_i defined by $A_i M$, where A_i is seen as an ideal of $A/\mathfrak{a}$. As $\mathfrak{m}_i$ is maximal, we have $A_{\mathfrak{m}_i}/(\mathfrak{m}_i A_{\mathfrak{m}_i})^{n_i} = A/\mathfrak{m}_i^{n_i}$. Therefore M_i is canonically a finitely generated non-zero torsion $A_{\mathfrak{m}_i}$-module. By the structure theorem for finitely generated non-zero torsion modules over a principal ideal domain (the proof of which is the same as for abelian groups) and because $A_{\mathfrak{m}_i}$ is a principal ideal domain with unique maximal ideal $\mathfrak{m}_i$ (see Theorem 9.3 in A-M), we thus find M_i isomorphic to $\prod_{1\leq k\leq k_i} A/\mathfrak{m}_i^{e_k}$ for unique positive integers k_i and $e_1, \ldots, e_{k_i}$. These facts sum up to the desired statement.