Commutative Algebra

D-MATH, HS 2014

Prof. Brent R. Doran

Solutions Sheet 12

1. The dimension of height of $\mathbb{C}[x_1, ..., x_s]_{\mathfrak{p}_r}$ equals the height of $\mathfrak{p}_r$. The latter is $\geq r$ because $0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ is a strictly increasing chain of prime ideals below $\mathfrak{p}_r$. On the other hand, Krull's Height Theorem (Corollary 11.16 in A-M) shows that the height of $\mathfrak{p}_r$ is $\leq r$. Thus $\dim \mathbb{C}[x_1, ..., x_s]_{\mathfrak{p}_r} = r$.

We indicate another approach which does not use Krull's Height Theorem. Let us first consider the dimension of $A = \mathbb{C}[x_1, \ldots, x_n]$. By definition we have that

$$\dim(A) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}})$$

where the supremum is over all maximal ideals $\mathfrak{m}$ of A. By the Nullstellensatz, every maximal ideal of A has the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in \mathbb{C}$. Hence all local rings are isomorphic and it follows that $\dim(A) = \dim(A_{\mathfrak{m}})$ where $\mathfrak{m} = (x_1, \ldots, x_n)$. Denote $B = A_{\mathfrak{m}}$ and let $\mathfrak{n} = \mathfrak{m}A_{\mathfrak{m}}$ be its maximal ideal.

Theorem 11.14 in A-M now implies that the dimension of B is equal to the degree of the characteristic polynomial $\chi_{\mathfrak{n}}(n) = l(B/\mathfrak{n}^n)$. But $l(B/\mathfrak{n}^n) = \sum_{d=0}^{n} l(\mathfrak{n}^{d-1}/\mathfrak{n}^d)$ (where $\mathfrak{n}^0 = B$). Each $\mathfrak{n}^{d-1}/\mathfrak{n}^d$ is a vector space over $B/\mathfrak{n} = \mathbb{C}$ with a basis given by the homogeneous polynomials of degree $d-1$ in n variables. But the dimension of the space of homogeneous polynomials of degree d in n variables is $\binom{n+d-1}{n-1}$ and hence the length of $\mathfrak{n}^{d-1}/\mathfrak{n}^d$ is a polynomial of degree $n-1$. This implies that $\chi_{\mathfrak{n}}(n)$ is of degree n and so $\dim(A) = n$.

Let $B = \mathbb{C}[x_1, \ldots x_n]_{\mathfrak{p}_r}$ where $\mathfrak{p}_r = (x_1, \ldots, x_r)$ and set $\mathfrak{n} := \mathfrak{p}_r B$. As above, we need to calculate the degree of $\chi_{\mathfrak{n}}(n)$. Note that in this case $\mathfrak{n}^{d-1}/\mathfrak{n}^d$ is a vector space over $B/\mathfrak{n} = \mathbb{C}(x_{r+1}, \ldots, x_n)$ with a basis given by the homogeneous polynomials of degree $d-1$ in the variables $x_1, \ldots, x_r$. As above it follows that $\dim(B) = r$.

2. The only one–dimensional k–algebra is k.

Suppose that A is of dimension 2 over k and choose $x \in A \setminus k$. Then $k[x] = A$ and x has to satisfy a quadratic polynomial p over k. As k is algebraically closed, we know that p decomposes into two linear factors. If they are different, we have that $A \simeq k[X]/(X-a)(X-b) = k \times k$. Otherwise, $A \simeq k[X]/(X-a)^2 \simeq k[X]/X^2$. These are 2 non–isomorphic types, because the algebra is reduced in the first but not in the second case.

Finally suppose that A is of dimension 3 over k. An elementary approach as above is still possible but tedious. So let us open the toolbox.

Using Theorem 8.7 in A-M we may restrict ourselves to the case where A is local. By Theorem 8.5 the maximal ideal $\mathfrak{m}$ is the unique prime ideal of A, hence nilpotent. By Nakayama $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ if $\mathfrak{m}^n \neq 0$. Furthermore $A/\mathfrak{m} \simeq k$ as k is algebraically closed. So $\dim_k \mathfrak{m} = 2$ and for dimension reasons $\mathfrak{m}^3 = 0$. Let x, y be a k-basis of $\mathfrak{m}$.

In the first case $\mathfrak{m}^2 \neq 0$. Hence $\dim_k \mathfrak{m}^2 = 1$. Without loss of generality we may assume $y \in \mathfrak{m}^2$. As $\mathfrak{m}^2$ is generated by x^2, $xy = 0$ and $y^2 = 0$, we find $x^2 \neq 0$. This yields the injective homomorphism $k[X]/X^3 \to A, X \mapsto x$. As the dimension of A is 3, it is in fact an isomorphism.

In the second case $\mathfrak{m}^2 = 0$. Then $x^2 = xy = y^2 = 0$. This induces the surjective homomorphism

$$k[X,Y]/(X^2, XY, Y^2) \to A$$

which sends X to x and Y to y. As both rings are of dimension 3 over k, it is in fact an isomorphism.

Over non–algebraically closed field the question is much more difficult and of arithmetic nature. For example the fields $\mathbb{Q}(\sqrt{p})$ are all non–isomorphic quadratic extension of $\mathbb{Q}$ for different prime numbers p.

3. Let $x_1, \ldots, x_r$ be elements of M such that there images in $M \otimes_A k \cong M/\mathfrak{m}M$ form a basis over k. By Nakayama's Lemma (Corollary 2.7 in A-M) they therefore span M. Consider the induced short exact sequence

$$0 \longrightarrow N \longrightarrow A^{\oplus r} \longrightarrow M \longrightarrow 0.$$

We claim that $N = 0$. As K is flat (see Propositions 3.3 and 3.5 in A-M) and by an assumption of the exercise, we find that $N \otimes_A K = 0$. Let $S := A \setminus 0$. Again by Proposition 3.5 in A-M we have $N \otimes_A K \cong S^{-1}N$. As N injects into $S^{-1}N$, we indeed get $N = 0$. This yields the exercise.

4. Due to the hint, it is enough to show that $\mathfrak{a}_0 A_\mathfrak{m} = \mathfrak{a} A_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of A. If $\mathfrak{m}$ contains $\mathfrak{a}$, then this is true by construction of $\mathfrak{a}_0$. If $\mathfrak{m}$ does not contain $\mathfrak{a}$, we consider two cases: If $x_0 \notin \mathfrak{m}$, then x_0 becomes invertible in $A_\mathfrak{m}$. If $x_0 \in \mathfrak{m}$, then $x_j \notin \mathfrak{m}$ for some $1 \leq j \leq s$, so that x_j becomes invertible in $A_\mathfrak{m}$. In both cases we therefore have $\mathfrak{a}_0 A_\mathfrak{m} = A_\mathfrak{m} = \mathfrak{a} A_\mathfrak{m}$.

5. (i) We shall show that the ring $S^{-1}A$ satisfies the conditions of Exercise 4. Note that every prime ideal in $S^{-1}A$ corresponds to a prime ideal $\mathfrak{q}$ in A that does not meet S, i.e. $\mathfrak{q} \subset \bigcup_i \mathfrak{p}_i$. We want to show that $\mathfrak{q} \subset \mathfrak{p}_i$ for some i. Let $f \in \mathfrak{q}$ with $f \neq 0$ and consider the set of variables x_s that appear in f. This is a finite set, hence there exist $j_1, \ldots, j_n$ such that $x_s \in \bigcup_k \mathfrak{p}_{j_k}$ for all s. Assume there is a $g \in \mathfrak{q} \setminus (\bigcup_k \mathfrak{p}_{j_k})$.

Claim: $f + g \notin \sum_{l=1}^{\infty} \mathfrak{p}_l$.

To prove this write $f = \sum_i f_i$ for distinct monomials f_i and $g = \sum_i g_i$ for distinct monomials g_i. We have the following facts:

(a) As $g \notin \bigcup_k \mathfrak{p}_{j_k}$ and $g \in \mathfrak{p}_d$ for some d, every monomial of g_i contains a variable indexed by one of $m_d + 1, \ldots, m_{d+1}$. Every monomial f_i contains only the variables associated to $\mathfrak{p}_{j_k}$ ($k = 1, \ldots, n$)). Hence all the f_i, g_i are distinct.

(b) Let $h = \sum_i h_i$ be a decomposition of some element h into monomials h_i. Then $h_i \notin \mathfrak{p}_j$ implies $h \notin \mathfrak{p}_j$. This follows from the fact that the ideals $\mathfrak{p}_j$ are homogeneous in *every* variable x_k.

(c) For each j_k there exist an i such that $g_i \notin \mathfrak{p}_{j_k}$.

(d) Analogously, for each $j \notin \{j_1, \dots, j_n\}$, $f_i \notin \mathfrak{p}_j$.

This implies the claim. But this is a contradiction as $f + g \in \mathfrak{q} \subset \sum_{l=1}^{\infty} \mathfrak{p}_l$. Therefore we have $\mathfrak{q} \subset \bigcup_{k=1}^{n} \mathfrak{p}_{j_k}$. It follows now from the Prime Avoidance Lemma (Proposition 1.11 in A-M) that $\mathfrak{q} \subset \mathfrak{p}_m$ for some m.

From the above it follows that every maximal ideal $\mathfrak{m}$ of $S^{-1}A$ is of the form $S^{-1}\mathfrak{p}_i$ for some i. The localization of $S^{-1}A$ along $\mathfrak{m}$ is isomorphic to $A_{\mathfrak{p}_i}$. To see that $A_{\mathfrak{p}_i}$ is noetherian, note that it is isomorphic to the localization of the finitely generated K-Algebra $K[x_{m_i+1}, \dots, x_{m_{i+1}}]$ along the ideal $(x_{m_i+1}, \dots, x_{m_{i+1}})$, where $K = k(x_1, \dots, x_{m_i}, x_{m_{i+1}+1}, x_{m_{i+1}+2}, \dots)$ is the field of rational functions in the variables that are not among the generators of $\mathfrak{p}_i$. This is clearly noetherian. The second condition of Exercise 4 is now immediate. We conclude that $S^{-1}A$ is Noetherian.

(ii) The height of $S^{-1}\mathfrak{p}_i$ is equal to $\dim(A_{\mathfrak{p}_i})$. But we have seen above that $A_{\mathfrak{p}_i}$ is isomorphic to $K[y_1, \dots, y_n]_{(y_1,\dots,y_n)}$ for a field K and variables $y_1, \dots, y_n$ where $n = m_{i+1} - m_i$. It follows now from Exercise 1 that the height of $S^{-1}\mathfrak{p}_i$ is $m_{i+1} - m_i$ as desired.

6. i). For a graded $\mathbb{C}$-module M, define the shifted module $M(d)$ by $M(d)_n = M_{d+n}$. If $P(M,t)$ is the Poincare series of M, then $P(M(d),t) = t^{-d}P(M,t)$.

Now let $J_1 := (x^3 + y^3 + z^3) \subset A = \mathbb{C}[x,y,z]$. Then as a graded $\mathbb{C}$-module we have $J_1 \cong A(-3)$ and therefore

$$P(J_1,t) = t^3 P(A,t) = \sum_{n \geq 3} \binom{n-1}{2} t^n$$

where we use $P(A,t) = \sum_{n\geq 0} \binom{n+2}{2} t^n$. Consider the exact sequence

$$0 \longrightarrow J_1 \longrightarrow A \longrightarrow A/J_1 \longrightarrow 0.$$

As P is addivite, we have

$$P(A/J_1,t) = P(A,t) - P(J_1,t) = 1 + \sum_{n\geq 1} 3nt^n.$$

ii). Let $I_1 := (x^2 + y^2 + z^2)$ and consider the exact sequence

$$0 \longrightarrow I_1 \cap J_1 \longrightarrow I_1 \oplus J_1 \longrightarrow J_2 \longrightarrow 0$$

We have $I_1 \cap J_1 = ((x^3 + y^3 + z^3) \cdot (x^2 + y^2 + z^2))$ which is isomorphic to $A(-5)$ as a graded $\mathbb{C}$-module. Hence

$$\begin{aligned} P(J_2, t) &= P(I_1, t) + P(J_1, t) - P(I_1 \cap J_1, t) \\ &= P(A(-2), t) + P(A(-3), t) - P(A(-5), t) \\ &= t^2 + \sum_{n \geq 3} \frac{1}{2} \left(n^2 + 3n - 10\right) t^n \end{aligned}$$

and

$$P(A/J_2, t) = 1 + 3t + 5t^2 + \sum_{n \geq 3} 6t^n.$$