Commutative Algebra

D-MATH, HS 2014

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Solutions Sheet 2

1. Assume first that $V(\mathfrak{a})$ is irreducible. In particular $\mathfrak{a} \neq A$ as $V(A) = \emptyset$. Consider $f, g \in A$ with $fg \in \mathfrak{a}$. Then $V(\mathfrak{a}) \subset V((f)\cdot(g)) = V(f) \cup V(g)$. By irreducibility of $V(\mathfrak{a})$ we may assume without loss of generality $V(\mathfrak{a}) \subset V(f)$. Thus $(f) \subset \mathrm{rad}(\mathfrak{a})$. As $\mathfrak{a}$ is radical, we deduce $f \in \mathfrak{a}$. This proves one direction.

For the other direction assume that $\mathfrak{a}$ is prime. Therefore $\mathfrak{a} \in V(\mathfrak{a})$ and in particular $V(\mathfrak{a})$ is not empty. Consider a covering $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$ of $V(\mathfrak{a})$ by closed subsets associated to ideals $\mathfrak{b}, \mathfrak{c}$ of A. We need to show that $V(\mathfrak{a})$ equals at least one of these subsets. As $\mathfrak{a} \in V(\mathfrak{a})$ we may assume $\mathfrak{a} \in V(\mathfrak{b})$. This implies $\mathfrak{b} \subset \mathfrak{a}$ and hence $V(\mathfrak{a}) \subset V(\mathfrak{b})$.

The remaining claim follows from $\mathrm{Spec}(A) = V((0)) = V(\mathrm{nil}(A))$.

2. (a) By definition of $\overline{Y}$, any non-empty open subset of $\overline{Y}$ has non-empty intersection with Y. From this, the statement follows immediately.

(b) This is implied through Zorn's Lemma as follows: Let Y be an irreducible subset of X. Consider the set M of irreducible subsets of X which contain Y. It is partially ordered through the inclusion relation. We have $M \neq \emptyset$, as $Y \in M$. We claim that for any chain $(Y_i)_{i \in I}$ in M, the union $\tilde{Y} := \bigcup_{i \in I} Y_i$ is in M, and therefore is an upper bound for $(Y_i)_{i \in I}$. Indeed, assume U, U' are non-empty open subsets of $\tilde{Y}$. There must thus exist $i_1, i_2 \in I$ with $U \cap Y_{i_1} \neq \emptyset$ and $U' \cap Y_{i_2} \neq \emptyset$. Without loss of generality $Y_{i_2} \subset Y_{i_1}$. As Y_{i_1} is irreducible, we find $(U \cap Y_{i_1}) \cap (U' \cap Y_{i_1}) \neq \emptyset$ and in particular $U \cap U' \neq \emptyset$. Hence $\tilde{Y} \in M$. This guarantees, by Zorn's Lemma, a maximal element in M, which is precisely a maximal irreducible subset of X containing Y.

(c) They are closed by (a). They cover X because each element of X forms an irreducible subset and is therefore contained in an irreducible component by (b). The irreducible components of a Hausdorff space are the one point subsets.

(d) By Exercise 1, irreducible closed subsets of $\mathrm{Spec}(A)$ are in one-to-one correspondence with prime ideals of A. Maximal irreducible closed subsets thereby correspond to minimal prime ideals.

3. (a) Take $\mathfrak{p} \in Y$. Then $\varphi^*(\mathfrak{p}) \in X_f$ if and only if $f \notin \varphi^{-1}(\mathfrak{p})$ if and only if $\varphi(f) \notin \mathfrak{p}$.

(b) Let $\psi\colon B \to C$ be a ring homomorphism. We have to show, that $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$: For any $\mathfrak{p} \in \mathrm{Spec}\, A$ we have

$$\varphi^* \circ \psi^*(\mathfrak{p}) = \varphi^{-1}(\psi^{-1}(\mathfrak{p})) = (\psi \circ \varphi)^{-1}(\mathfrak{p}) = (\psi \circ \varphi)^*(\mathfrak{p}).$$

Moreover, we have $(\mathrm{id}_B)^* = \mathrm{id}_{\mathrm{Spec}(B)}$.

(c) Take $\mathfrak{p} \in \operatorname{Spec} B$. Hence $\varphi^*(\mathfrak{p}) \in V(\mathfrak{a})$ if and only if $\mathfrak{a} \subset \varphi^{-1}(\mathfrak{p})$ if and only if $\varphi(a) \subset \mathfrak{p}$. This is the case if and only if $\mathfrak{a}^e \subset \mathfrak{p}$.

(d) Take any $\mathfrak{p} \in V(\mathfrak{b})$. Then $\varphi^{-1}(\mathfrak{p}) \supset \mathfrak{b}^c$. Therefore $\varphi^*(V(\mathfrak{b})) \subset V(\mathfrak{b}^c)$ and $\overline{\varphi^*(V(\mathfrak{b}))} \subset V(\mathfrak{b}^c)$. For the other inclusion let

$$V(J) = \overline{\varphi^*(V(\mathfrak{b}))} \supset \varphi^*(V(\mathfrak{b}))$$

for an ideal $J \subset A$. So

$$\operatorname{rad} J = \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p} \supset \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \varphi^{-1}(\mathfrak{q}) = \varphi^{-1}\left(\bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q} \right) = \varphi^{-1}(\operatorname{rad} \mathfrak{b}) = \operatorname{rad} \mathfrak{b}^c.$$

Hence $V(\mathfrak{b}^c) \subset V(J)$.

(e) In this case, $B \cong A/\ker(\varphi)$. The projection homomorphism $A \to A/\ker(\varphi)$ induces a bijection between ideals of $A/\ker(\varphi)$ and ideals of A containing $\ker(\varphi)$ which preserves inclusions. The restriction of this bijection to prime ideals is therefore a homoemorphism.

(f) Using exercise (d) with $\mathfrak{b} = 0$, we see that $\overline{\varphi^*(Y)} = V(\ker(\varphi))$. Hence $\varphi^*(Y)$ is dense if and only if $\ker(\varphi) \subset \mathfrak{N}$.

(g) Recall that for arbitrary rings C and D the spectrum $\operatorname{Spec}(C \times D)$ is homeomorphic to the disjoint union $\operatorname{Spec} C \coprod \operatorname{Spec} D$. As both $\operatorname{Spec} A/\mathfrak{p}$ and $\operatorname{Spec} K$ are one-point spaces, we thus see that $\operatorname{Spec} B$ is homeomorphic to the discrete space with two elements. This has four closed subsets. On the other hand, the closed subsets of $\operatorname{Spec} A$ are precisely $\emptyset$, $\{\mathfrak{p}\}$ and $\operatorname{Spec} A$, where $\mathfrak{p}$ denotes the non-zero prime ideal of A. From this we already see that there exists no homeomorphism between $\operatorname{Spec} B$ and $\operatorname{Spec} A$.

Explicitely, the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ assigns

$$\{0\} \times K \mapsto \mathfrak{p}$$

and

$$A/\mathfrak{p} \times \{0\} \mapsto \{0\}.$$

So it is a bijection. It sends the closed point $A/\mathfrak{p} \times \{0\}$ to the non-closed point $\{0\}$.

4. We show more generally for any ideals I, J of a ring A that

$$A/I \otimes A/J \simeq A/(I+J).$$

In fact, applying Exercise 5 to $\mathfrak{a} := I$ and $M := A/J$ already yields

$$A/I \otimes_A A/J \cong (A/J)/I(A/J) \cong (A/J)/(I+J/J) \cong A/(I+J).$$

We indicate another approach to this exercise: We use the fact that in both the category of algebras over a fixed ring and the category of modules over a fixed ring, to show that any objects C, D are isomorphic, it is enough to show that there is an isomorphism

$$\operatorname{Hom}(C, \_) \simeq \operatorname{Hom}(D, \_)$$

of functors. The latter means that we have for any test object E an isomorphism

$$\mathrm{Hom}(C, E) \simeq \mathrm{Hom}(D, E)$$

which is natural in E. This fact is known as the Yoneda-Lemma (see for instance [*Awodey, Category theory*]).

Let finally I, J again be ideals of a ring A. The ring homomorphism $A/I \to A/I \otimes_A A/J$, which sends an element $\overline{a}$ to $\overline{a} \otimes 1$, turns $A/I \otimes_A A/J$ into an A/I-algebra. To show that, as an A-module, it is isomorphic to $A/(I+J)$ take any A/I–algebra B. Then

$$\begin{aligned}\mathrm{Hom}_{A/I\text{–alg}}(A/I \otimes_A A/J, B) &= \mathrm{Hom}_{A\text{–alg}}(A/J,\ B) = \mathrm{Hom}_{A\text{–alg}}(A/(I+J), B)\\ &= \mathrm{Hom}_{A/I\text{–alg}}(A/(I+J), B)\end{aligned}$$

By the Yoneda–Lemma we therefore get an isomorphism $A/I \otimes_A A/J \simeq A/(I+J)$ of A/I–algebras and thus also of A–algebras and of A-modules.

5. Consider the short exact sequence of A-modules $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$. As tensoring with the A-module M is right-exact, we get the exact sequence $\mathfrak{a} \otimes_A M \to A \otimes_A M \to (A/\mathfrak{a}) \otimes_A M \to 0$. Canonically, $A \otimes_A M \cong M$. The image of $\mathfrak{a} \otimes_A M$ in M is precisely $\mathfrak{a}M$. This yields $(A/\mathfrak{a}) \otimes_A M \cong M/\mathfrak{a}M$.

6. Let ψ respectively ϕ denote the homomorphism from M' to M respectively from M to M''. Let T' be a finite subset of M' which generates M' and let T'' be a finite subset of M'' which generates M''. As ϕ is surjective we can choose for any element of T'' a preimage in M. Let S be the finite set of these chosen preimages. From $\mathrm{Im}(\psi) = \ker(\phi)$ we see that $\psi(T') \cup S$ generates M.