Commutative Algebra

**D-MATH, HS 2014**

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# Solutions Sheet 5

**1.** The irreducible ideals of $\mathrm{Spec}(A/\mathfrak{a})$ correspond to the minimal prime ideals above $\mathfrak{a}$. By Proposition 4.6 the latter correspond bijectively to the minimal prime ideals belonging to $\mathfrak{a}$. Thus the statement follows from the finiteness of the set of prime ideals belonging to $\mathfrak{a}$.

**2.** By the solutions to Exercise 4, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals and therefore primary. Moreover, $\mathfrak{m}$ is a maximal ideal. By Proposition 4.2, any power of $\mathfrak{m}$ is primary. We show that $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. We have $\mathfrak{p}_1\mathfrak{p}_2 = (x^2, xy, xz, yz)$ and $\mathfrak{m}^2 = (x^2, y^2, z^2, xy, xz, yz)$. From this we see immediately that $\mathfrak{p}_1\mathfrak{p}_2 \subset \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. To see the converse inclusion, we first show $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, yz)$: Let $\lambda x + \mu y \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\lambda, \mu \in k[x, y, z]$. Consequently $\mu y \in (x, z)$. Therefore $\mu \in (x, z)$ as this ideal is prime and does not contain $y$. Writing $\mu$ in terms of the generators $x$ and $z$, we see that $\mu y \in (x, yz)$. This shows $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset (x, yz)$. The converse inclusion is immediate. To see $\mathfrak{p}_1\mathfrak{p}_2 \supset (x, yz) \cap \mathfrak{m}^2$ we first observe that $x^2, xy, xz, yz \in \mathfrak{p}_1\mathfrak{p}_2 \cap (x, yz)$. It is thus enough to show that any element of the form $ay^2 + bz^2 \in (x, yz) \cap \mathfrak{m}^2$, where $a, b \in k[x, y, z]$, is also contained in $\mathfrak{p}_1\mathfrak{p}_2$. We may write $ay^2 + bz^2 = cx + dyz$ for further $c, d \in k[x, y, z]$. Thus, $cx \in (y, z)$ and hence $c \in (y, z)$, as this ideal is prime. Writing $c$ in terms of the the generators $y$ and $z$, we find $cx \in \mathfrak{p}_1\mathfrak{p}_2$. As also $dyz \in \mathfrak{p}_1\mathfrak{p}_2$, we have $ay^2 + bz^2 \in \mathfrak{p}_1\mathfrak{p}_2$. This shows the desired inclusion.
It remains to show that no one of the components in this primary decomposition of $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is contained in the intersection of the others. First, we have $\mathfrak{p}_1 \cap \mathfrak{m}^2 \not\subset \mathfrak{p}_2$, since otherwise, by Proposition 1.11, $\mathfrak{p}_1 \subset \mathfrak{p}_2$ or $\mathfrak{m}^2 \subset \mathfrak{p}_2$, which would yield a contradiction. By the same argument, also $\mathfrak{p}_2 \cap \mathfrak{m}^2 \not\subset \mathfrak{p}_1$. Assume finally, by contradiction, that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subset \mathfrak{m}^2$ and in particular that $x \in \mathfrak{m}^2$. Then, by symmetry, also $y, z \in \mathfrak{m}^2$ and thus $\mathfrak{m} = \mathfrak{m}^2$. Applying Nakayama's Lemma [Proposition 2.4] we find $r \in \mathfrak{m}$ with $(1 + r)\mathfrak{m} = 0$ yielding a contradiction to $\mathfrak{m} \neq 0$.

**3.** (i) This is straightforward.

(ii) Consider for any ideal $\mathfrak{a} \subset A$ the surjective ring homomorphism $A[x] \to (A/\mathfrak{a})[x]$ which reduces the coefficients of elements in $A[x]$ modulo $\mathfrak{a}$. It has $\mathfrak{a}$ as its kernel. If $\mathfrak{p}$ is prime, then $(A/\mathfrak{p})$ is an integral domain, and hence also $(A/\mathfrak{p})[x]$. Consequently, in this case $\mathfrak{p}[x]$ is a prime ideal.

(iii) Using the homomorphism from (ii) we find $A[x]/\mathfrak{q}[x] \cong (A/\mathfrak{q})[x]$. We thus need to show that any zero divisor $f \in (A/\mathfrak{q})[x]$ is nilpotent. By Exercise 1,(iii) of the first exercise sheet, there exists an $a \in A/\mathfrak{q}$ such that $af = 0$,

i.e. $af_k = 0$ for any coefficient $f_k$ of $f$. As $\mathfrak{q}$ is primary, this implies, that any of the coefficients $f_k$ is nilpotent. By Exercise 1,(iv) of the first exercise sheet, this implies that $f$ is nilpotent.
To see that the radical of $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$ it is enough to show that the nil radical of $(A/\mathfrak{q})[x]$ is $(\mathfrak{p}/\mathfrak{q})[x]$. The latter follows again from Exercise 1,(iv) of the first exercise sheet and the assumption that $\mathfrak{p}/\mathfrak{q}$ is the nil radical of $A/\mathfrak{q}$.

(iv) The equality of sets $\mathfrak{a}[x] = \bigcap_{1\leq i\leq n} \mathfrak{q}_i[x]$ is checked immediately. By (ii) it is a primary decomposition with associated prime ideals $(\mathfrak{p}_i[x])_{1\leq i\leq n}$, where $(\mathfrak{p}_i)_{1\leq i\leq n}$ are the prime ideals belonging to $\mathfrak{a}$. The $\mathfrak{p}_i[x]$ are different from one another, as the $\mathfrak{p}_i$ are. For any ideal $\mathfrak{b}$ of $A$ the contraction of the extension of $\mathfrak{b}$ under the homomorphism $A \to A[x]$ equals $\mathfrak{b}$. By this and the assumption we see that none of the $\mathfrak{q}_i[x]$ contains the intersection of all the others. Hence the decomposition is minimal.

(v) Parts (iii) and (iv) set up an inclusion preserving bijection $\mathfrak{p} \mapsto \mathfrak{p}[x]$ between the prime ideals belonging to $\mathfrak{a}$ and the prime ideals belonging to $\mathfrak{a}[x]$. The minimal prime ideals belonging to $\mathfrak{a}$ are therefore associated to minimal prime ideals belonging to $\mathfrak{a}$. By Proposition 4.6, these are precisely the minimal prime ideals of $\mathfrak{a}$ respectively of $\mathfrak{a}[x]$.

**4.** We show this by induction on $n$. If $n = 1$, then $(x_1) \subset k[x_1]$ is a maximal ideal. By Proposition 4.2, its powers are therefore primary. If $n > 1$ and $1 \leq i \leq n$, consider $(x_1, \ldots, x_{i-1}) \subset k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, which we set to be zero if $i = 1$. By assumption, this is prime . Hence also $(x_1, \ldots, x_i) = (x_1, \ldots, x_{i-1})[x_i] \subset k[x_1, \ldots, x_n]$ is prime by Exercise 3, (ii). To see that its powers are primary we distinguish two cases. The first is $i = n$: Then $(x_1, \ldots, x_n)$ is a maximal ideal and we conclude again by Proposition 4.2. The second is $i < n$: Then we have for any $m \geq 1$ that $(x_1, \ldots, x_i)^m = (x_1, \ldots, x_i)^m[x_n]$, where on the left hand side we have considered $(x_1, \ldots, x_i)^m$ as an ideal of $k[x_1, \ldots, x_n]$ and on the right hand side as an ideal of $k[x_1, \ldots, x_{n-1}]$. By inductive hypothesis it is a primary ideal in the latter case. By Exercise 3,(iii) we find that $(x_1, \ldots, x_i)^m$, considered as an ideal of $k[x_1, \ldots, x_n]$, is primary.

**5.** (i) The extension $\mathfrak{p}A_\mathfrak{p}$ of $\mathfrak{p}$ is a maximal ideal. Hence, by Proposition 4.2, the power $(\mathfrak{p}A_\mathfrak{p})^n$ is $\mathfrak{p}A_\mathfrak{p}$-primary. Hence its contraction $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.

(ii) We apply Proposition 4.9 to $S := A\backslash\mathfrak{p}$. Let $\mathfrak{p}^n = \bigcap_{1\leq i\leq k} \mathfrak{q}_i$ be a primary decomposition and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ denote the prime ideals belonging to $\mathfrak{p}^n$. Hence $\mathfrak{p} \subset \mathfrak{p}_i$ for any $1 \leq i \leq k$. Such an inclusion is strict if and only if $\mathfrak{p}_i$ meets $S$. By the Proposition, $\mathfrak{p}^{(n)}$ equals the intersection of all the $\mathfrak{q}_i$ for which $\mathfrak{p}_i$ does not meet $S$. In particular there exists a unique $\mathfrak{q}_i$ which does not meet $S$. Consequently, its radical equals $\mathfrak{p}$. This shows that $\mathfrak{p}$ appears in the prime ideals belonging to $\mathfrak{p}^n$ and that in any minimal primary decomposition of $\mathfrak{p}^n$ the $\mathfrak{p}$-primary component is $\mathfrak{p}^{(n)}$.

(iii) If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, then $\mathfrak{p}^n$ is $\mathfrak{p}$-primary by part (i). The converse statement follows immediately from part (ii).

**6.** The kernel $S_\mathfrak{p}(0)$ consists precisely of the elements $x \in A$ for which there exists an $s \in A\backslash\mathfrak{p}$ with $sx = 0$. Consider a $\mathfrak{p}$-primary ideal $\mathfrak{a}$. For any such $x$ and $s$ we

have $sx \in \mathfrak{a}$ and $s$ is not contained in the radical $\mathfrak{p}$ of $\mathfrak{a}$. Thus, as $\mathfrak{a}$ is primary, we find $x \in \mathfrak{a}$.
For the second part we refer to the proof given in Exercise 19 of Chapter 4 in Atyiah MacDonalds book.